Read Me First: *Show all essential work very neatly. Use correct notation when presenting your computations and arguments. Write using complete sentences. Be careful. Remember this: "=" denotes "equals" , "⇒" denotes "implies" , and "⇔" denotes "is equivalent to". Do not "box" your answers. Communicate. Show me the all magic on the page.*

1. (7 pts.) Locate and classify the singular points of the following second order homogeneous O.D.E. Use complete sentences to describe the type of points and where they occur.

$$((x-2)^2(x-1)^2)y'' + 4x(x-2)y' + (x-1)y = 0$$

2. (18 pts.) For parts (a), (b), and (c) below pretend that $x_0 = 2$ is a regular singular point for some homogeneous linear differential equation of the form $y''(x) + P_1(x)y'(x) + P_2(x)y(x) = 0$, with the ODE actually being different for each part. For each part, given the indicial equation at $x_0 = 2$ provided, use all the information available and Theorem 6.3 to say what the two nontrivial linearly independent solutions look like without attempting to obtain the coefficients of the power series involved.

(a) Indicial equation: $(r - (1/3))(r - (1/2)) = 0$

$y_1(x) =$

$y_2(x) =$

(b) Indicial equation: $(r - (1/2))(r - (1/2)) = 0$

$y_1(x) =$

$y_2(x) =$

(c) Indicial equation: $(r - (3/2))(r + (7/2)) = 0$

$y_1(x) =$

$y_2(x) =$

3. (15 pts.) Very carefully obtain the solution to the following initial value problem that involves an Euler-Cauchy equation.

$$(*)\begin{cases} x^2y''(x) + xy'(x) + 4y = 4\ln(x) + 8 \ ; \\ y(1) = 1 \ , \ \ y'(1) = -1 \ \ \textit{with } x > 0. \end{cases}$$

4. (10 pts.) (a) Suppose that f(t) is defined for t > 0. What is the definition of the Laplace transform of f, $\mathfrak{L}\{f(t)\}$, in terms of a definite integral??

$\mathfrak{L}\{f(t)\}(s) =$

(b) Using only the definition, not the table, compute the Laplace transform of

$$f(t) = \begin{cases} 0 \ , \ \textit{if} \ \ 0 < t < 4 \\ 1 \ , \ \textit{if} \ \ 4 < t. \end{cases}$$

$\mathfrak{L}\{f(t)\}(s) =$

5. (15 pts.) Suppose

$$y(x) = \sum_{n=0}^{\infty} c_n x^n$$

is a solution of the homogeneous second order linear equation

$$y'' - y' + 2xy = 0.$$

Obtain the recurrence formula(s) for the coefficients of $y(x)$.

6. (10 pts.)
Compute $f(t) = \mathcal{L}^{-1}\{F(s)\}(t)$ when

(a) $F(s) = \dfrac{4}{s^4} + \dfrac{3s+4}{s^2+3}$

(b) $F(s) = \dfrac{8s + 7}{(s-2)^2 + 9}$

7. (5 pts.) The equation below has a regular singular point at $x_0 = 0$.

$$x^2y'' + 2xy' + (x^2 - 6)y = 0$$

Obtain the indicial equation at $x_0 = 0$, and determine its roots.

8. (10 pts.) The solution to a certain linear ordinary differential equation with coefficient functions that are analytic at $x_0 = 0$ is of the form

$$y(x) = \sum_{n=0}^{\infty} c_n x^n$$

where the coefficients for $n \geq 2$ satisfy the following equations:

$$(n+2)(n+1)c_{n+2} - n(n+1)c_{n+1} + c_n = 0 \quad \textit{for all } n \geq 1, \quad \textit{and} \quad c_2 = -\frac{1}{2}c_0.$$

Determine the coefficients c_0, c_1, c_2, c_3, and c_4 for the particular solution that satisfies the initial conditions $y(0) = 0$ and $y'(0) = 1$

9. (10 pts.) Transform the given initial value problem into an algebraic equation in $\mathfrak{L}\{y\}$ and solve for $\mathfrak{L}\{y\}$. Do not take inverse transforms and do not attempt to combine terms over a common denominator. Be very careful.

$$y''(x) + 3y'(x) + 2y = 0 \ ; \ \ y(0) = 1 \ , \ \ y'(0) = 2 \ .$$

Silly 10 Point Bonus: Compute the Laplace transform of the function

$$f(t) = \sin^3(bt).$$

Say where your work is, for it won't fit here.