Read Me First: *Show all essential work very neatly. Use correct notation when presenting your computations and arguments. Write using complete sentences. Be careful. Remember this: "=" denotes "equals" , "⇒" denotes "implies" , and "⇔" denotes "is equivalent to". Do not "box" your answers. Communicate. Show me the all magic on the page.*

1. (15 pts.) (a) If $f(t)$ and $g(t)$ are piecewise continuous functions defined for $t \geq 0$, what is the definition of the convolution of f with g, $(f*g)(t)$??

$$(f*g)(t) = \int_0^t f(x)g(t-x)\,dx$$

(b) Using only the definition of the convolution as a definite integral, not some fancy transform shenanigans, compute $(f*g)(t)$ when $f(t) = e^{4t}$ and $g(t) = e^{2t}$.

$$(f*g)(t) = \int_0^t f(x)g(t-x)\,dx = \int_0^t e^{4x}e^{2(t-x)}\,dx$$

$$= e^{2t}\int_0^t e^{2x}\,dx = e^{2t}\left[\left(\frac{e^{2x}}{2}\right)\Bigg|_0^t\right]$$

$$= e^{2t}\left[\frac{e^{2t}}{2} - \frac{1}{2}\right] = \frac{e^{4t} - e^{2t}}{2}.$$

(c) Using the Laplace transform table, compute the Laplace transform of $f*g$ when $f(t) = t\cdot\cos(t)$ and $g(t) = t^2e^{-2t}$. [Do not attempt to simplify the algebra after computing the transform.]

$$\mathcal{L}\{(f*g)(t)\}(s) = \mathcal{L}\{f(t)\}(s)\mathcal{L}\{g(t)\}(s)$$
$$= \mathcal{L}\{t\cos(t)\}\,\mathcal{L}\{t^2e^{-2t}\}$$
$$= \frac{s^2-1}{(s^2+1)^2}\cdot\frac{2!}{(s+2)^3}.$$

2. (10 pts.) Suppose that the Laplace transform of the solution to a certain initial value problem involving a linear differential equation with constant coefficients is given by

$$\mathcal{L}\{y(t)\}(s) = \frac{10s\,e^{-2\pi s}}{s^2+25} + \frac{4s+8}{s^2+2s+5}.$$

What's the solution, $y(t)$, to the IVP??

$$y(t) = 10\,u_{2\pi}(t)\mathcal{L}^{-1}\left\{\frac{s}{s^2+25}\right\}(t-2\pi) + \mathcal{L}^{-1}\left\{\frac{4s+8}{(s+1)^2+2^2}\right\}(t)$$

$$= 10\,u_{2\pi}(t)\cos(5(t-2\pi)) + 4\mathcal{L}^{-1}\left\{\frac{(s+1)+1}{(s+1)^2+2^2}\right\}(t)$$

$$= 10\,u_{2\pi}(t)\cos(5t) + 4e^{-t}\cos(2t) + 2e^{-t}\sin(2t).$$

Obviously, you may expand y into a piecewise-defined varmint.

3. (40 pts.) Without evaluating any integrals and using only the table provided, properties of the Laplace transform, and appropriate function identities, obtain the Laplace transform of each of the functions that follows. (8 pts./part)

(a) $f(t) = \begin{cases} 1 \ , \ if \ \ 0 < t < 1 \\ -5 \ , \ if \ \ 1 < t < 2 \\ 3 \ , \ if \ \ 2 < t. \end{cases}$

Writing f in terms of our friendly unit step functions, we have

$$\begin{aligned} \text{f}(t) &= 1u_0(t) + ((-5)-(1))u_1(t) + (3-(-5))u_2(t) \\ &= 1u_0(t) - 6u_1(t) + 8u_2(t). \end{aligned}$$

Thus, from the linearity of the Laplace transform, we have

$$\mathfrak{L}\{\text{f}(t)\}(s) = \frac{1}{s} - \frac{6}{s}e^{-s} + \frac{8}{s}e^{-2s}$$

(b) $g(t) = 3te^{4t}\sin(2t)$

$$\begin{aligned} \mathfrak{L}\{g(t)\}(s) &= 3\mathfrak{L}\{e^{4t}(t\sin(2t))\}(s) = 3\mathfrak{L}\{t\sin(2t)\}(s-4) \\ &= \frac{3[4(s-4)]}{((s-4)^2+4)^2}. \end{aligned}$$

This transform may also be obtained by following a line of reasoning that begins with

$$\mathfrak{L}\{g(t)\}(s) = 3\mathfrak{L}\{t(e^{4t}\sin(2t))\}(s) = -3\frac{d}{ds}\mathfrak{L}\{e^{4t}\sin(2t)\}(s) = \ldots.$$

Obviously, the second route is messier.

(c) $h(t) = \cos^2(t)e^{2t}$

$$\begin{aligned} \mathfrak{L}\{h(t)\}(s) &= \mathfrak{L}\{\cos^2(t)e^{2t}\}(s) \\ &= \mathfrak{L}\left\{\left(\frac{1+\cos(2t)}{2}\right)e^{2t}\right\}(s) \\ &= \frac{1}{2(s-2)} + \frac{s-2}{2\left((s-2)^2+4\right)}. \end{aligned}$$

(d) $f(t) = 10\cdot\delta(t - 8\pi)$

$$\mathfrak{L}\{f(t)\}(s) = 10\mathfrak{L}\{\delta(t-8\pi)\} = 10e^{-8\pi s}.$$

3. (40 pts.) Without evaluating any integrals and using only the table provided, properties of the Laplace transform, and appropriate function identities, obtain the Laplace transform of each of the functions that follows. (8 pts./part)

(e)

$$g(t) = \begin{cases} 2t \ , & \text{if} \quad 0 < t < 3 \\ 6 \ , & \text{if} \quad 3 < t. \end{cases}$$
$$= 2t + (6-2t)u_3(t)$$

$$\begin{aligned} \mathcal{L}\{g(t)\}(s) &= \mathcal{L}\{2t\}(s) + \mathcal{L}\{(6-2t)u_3(t)\}(s) \\ &= \frac{2}{s^2} + \mathcal{L}\{h(t-3)u_3(t)\}(s), \text{ where } h(t-3) = 6-2t \\ &= \frac{2}{s^2} + e^{-3s}\mathcal{L}\{h(t)\}, \text{ where } h(t) = h((t+3)-3) = -2t \\ &= \frac{2}{s^2} + e^{-3s}\mathcal{L}\{-2t\}(s) = \frac{2}{s^2} - e^{-2s}\left(\frac{2}{s^2}\right) \end{aligned}$$

4. (10 pts.) (a) The Laplace transform of the following periodic function may be written in terms of a definite integral. Simply express the transform in terms of the appropriate definite integral, but do not attempt to evaluate that definite integral.

$$f(t) = \begin{cases} t^2 & , \text{ if } \quad 0 \le t < 2 \\ (4-t)^2 & , \text{ if } \quad 2 \le t < 4, \\ f(t+4) = f(t) & , \text{ for all } t \ge 0. \end{cases}$$

$$\mathcal{L}\{f(t)\}(s) = \frac{\int_0^4 f(t)e^{-st}dt}{1 - e^{-4s}}$$

(b) The following sum of definite integrals can be realized as the Laplace transform of a certain function $g(t)$ defined for
$t \ge 0$. Provide the precise definition of that function g.

$$\int_0^2 t^2 e^{-st}dt + \int_2^4 (t-4)^2 e^{-st}dt = \mathcal{L}\{g(t)\}(s), \quad \text{where}$$

$$g(t) = \begin{cases} t^2 & , \text{ if } \quad 0 \le t < 2 \\ (4-t)^2 & , \text{ if } \quad 2 \le t < 4, \\ 0 & , \text{ if } \quad 4 \le t \ . \end{cases}$$

Warning: *Do not attempt to evaluate the Laplace transform of g.*

---

5. (10 pts.) Very neatly transform the given initial value problem into a linear system in $\mathcal{L}\{x\}$ and $\mathcal{L}\{y\}$ and stop. Do not attempt to solve for $\mathcal{L}\{x\}$ or $\mathcal{L}\{y\}$.

$$\begin{cases} x'(t) + y(t) = 0 \\ x(t) - y'(t) = 0, \end{cases}$$
$$x(0) = 3 \;\; and \;\; y(0) = -2.$$

By taking the Laplace Transform of both sides of the differential equations, using the initial conditions, and doing a little trivial algebra in our heads, we have

$$\begin{cases} s\mathcal{L}\{x\} + \mathcal{L}\{y\} = 3 \\ \mathcal{L}\{x\} - s\mathcal{L}\{y\} = 2 . \end{cases}$$

---

6. (15 pts.) Using only the Laplace transform machine, very carefully solve the following very dinky first order initial value problem:

$$y'(t) - y(t) = e^t\cos(4t) \;\; ; \; y(0) = 1.$$

By taking the Laplace Transform of both sides of the differential equation, and using the initial condition, we have

$$\mathcal{L}\{y'(t)\}(s) - \mathcal{L}\{y(t)\}(s) = \mathcal{L}\{e^t\cos(4t)\}$$

$$\Rightarrow \quad s\mathcal{L}\{y(t)\}(s) - y(0) - \mathcal{L}\{y(t)\}(s) = \frac{s-1}{(s-1)^2+16}$$

$$\Rightarrow \quad (s-1)\mathcal{L}\{y(t)\}(s) = 1 + \frac{s-1}{(s-1)^2+16}$$

$$\Rightarrow \quad \mathcal{L}\{y(t)\}(s) = \frac{1}{s-1} + \frac{1}{(s-1)^2+16}.$$

By taking inverse transforms now, we quickly obtain

$$y(t) = e^t + \frac{1}{4}e^t\sin(4t).$$

Silly 10 Point Bonus: If you hold your mouth just right and squint just so, you can evaluate the following improper integral with less than ten pages of work:

$$\int_0^\infty \left(\int_0^t \sin(2x)(t-x)^5 \, dx \right) e^{-t} \, dt$$

Say where your work is, for it won't fit here.

$$\int_0^\infty \left(\int_0^t \sin(2x)(t-x)^5 \, dx \right) e^{-t} \, dt = \frac{2}{5} \times \frac{5!}{1} = 48$$

after becoming one with convulsive laughter at convoluted reasoning??
See Test 4, Spring 2002 for the details of a similar bit of silliness.