NAME:

Student Number: Exam Number:

Read Me First: *Read each problem carefully and do exactly what is requested. Full credit will be awarded only if you show all your work neatly, and it is correct. Use complete sentences and use notation correctly. Remember that what is illegible or incomprehensible is worthless. Communicate. Good Luck!*

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1. (100 pts.) Solve each of the following differential equations or initial value problems. If an initial condition is not given, display the general solution to the differential equation.
(20 pts./part)

(a) $\frac{dr}{d\theta} + \tan(\theta)r = 2\cos^2(\theta)$

(b) $5\frac{dy}{dx} + \frac{y}{x} = 20x^3y^{-4}$ *with* $y(1) = 2$.

1. (Continued) Solve each of the following differential equations or initial value problems. If an initial condition is not given, display the general solution to the differential equation.
(20 pts./part)

(c) $(4y^2 + xy + x^2)dx - (x^2)dy = 0$

(d) $\dfrac{d^2y}{dx^2} + y = \cot(x)$

1. (Continued) Solve each of the following differential equations or initial value problems. If an initial condition is not given, display the general solution to the differential equation.
(20 pts./part)

(e) $\left(y^2 e^{2x}\right) dx + \left(y e^{2x} + 2y\right) dy = 0$

2. (20 pts.) (a) If

$$f(t) = \begin{cases} -1 & , \ if \quad 0 < t < 3 \\ 2t - 7 & , \ if \quad 3 < t < 5 \\ 3 & , \ if \quad 5 < t , \end{cases}$$

then $\mathfrak{L}\{f(t)\}(s) =$

(b) Write $f(t) = \mathfrak{L}^{-1}\{F(s)\}(t)$ in piecewise defined form when

$$F(s) = \frac{4s - 5}{s^2 + 4} e^{-\pi s}.$$

$f(t) = \mathfrak{L}^{-1}\{F(s)\}(t) =$

3. (30 pts.) Solve each of the following first order initial-value problems using only the Laplace transform machine.

(a) $y'(t) - y(t) = e^{t}\cos(4t)$; $y(0) = 2\pi$.

(b)

$$\begin{cases} x'(t) + 2y(t) = \cos(2t) \\ -2x(t) + y'(t) = \sin(2t) \end{cases},$$

$x(0) = 0$ *and* $y(0) = 0$.

4. (15 pts.) (a) Obtain the differential equation satisfied by the family of curves defined by the equation (*) below.

(b) Next, write down the differential equation that the orthogonal trajectories to the family of curves defined by (*) satisfy.

(c) Finally, solve the differential equation of part (b) to obtain the equation(s) defining the orthogonal trajectories. [These, after all, are another family of curves.]

$$(*) \quad y = cx^2.$$

5. (10 pts.) The equation

$$xy'' - y' + xy = 0$$

has a regular singular point at $x_0 = 0$. (a) Find the indicial equation of this O.D.E. at $x_0 = 0$ and determine its roots. (b) Then, using all the information now available and Theorem 6.3, say what the general solution at $x_0 = 0$ looks like without attempting to obtain the coefficients of the power series functions involved.

6. (15 pts.) (a) (5 pts.) Obtain the recurrence formula(s) satisfied by the coefficients of the power series solution y at $x_0 = 0$, an ordinary point of the homogeneous ODE

$$y'' - 2xy = 0.$$

(b) (5 pts.) Compute the first five (5) coefficients of the power series solution y_1 that satisfies the initial conditions $y(0) = 1$ and $y'(0) = -1$.
(c) (5 pts.) Suppose y_2 is the unique solution to the ODE which satisfies the initial conditions $y(0) = 1$ and $y'(0) = 2$. Determine, with proof, whether y_1 from Part b, and y_2 are linearly independent.

7. (5 pts.) Obtain the initial-value problem needed to solve the following problem which uses Hooke's law but do not attempt to solve the differential equation or initial-value problem you obtain. **Be sure to state what your variables represent using complete sentences.**

//An eight pound weight is attached to the lower end of a coil spring suspended from a fixed support. The weight comes to rest in its equilibrium position, thereby stretching the spring 6 inches. The weight is then pulled down 9 inches below its equilibrium position and released at t = 0. The medium offers a resistance in pounds numerically equal to 4x′, where x′ is the instantaneous velocity in feet per second. Assuming there are no externally impressed forces, determine the displacement of the weight as a function of time.//

8. (5 pts.) Circle the letter corresponding to the correct response: If $F(s) = \mathfrak{L}\{t\sin(bt)e^{at}\}(s)$,then $F(s) =$

(a) $\left[\frac{1}{s^2}\right]\cdot\left[\frac{b}{s^2+b^2}\right]\cdot\left[\frac{1}{s-a}\right]$ (b) $\frac{d}{ds}\left[\frac{b}{(s-a)^2+b^2}\right]$

(c) $\frac{2b(s-a)}{((s-a)^2+b^2)^2}$ (d) $\left[\frac{2bs}{(s^2+b^2)^2}\right]\cdot\left[\frac{1}{s-a}\right]$

(e) $\left[\frac{b}{s^2+b^2}\right]\cdot\left[\frac{1}{(s-a)^2}\right]$ (f) None of (a) through (e).

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10 Screaming Yellow Bonkers Bonus Points:

What’s the exact value of the following definite integral

$$\int_0^\infty |\cos(t)|e^{-t}\,dt \text{ ??}$$

Hints:
(a) Yes, this may be viewed in terms of the Laplace transform.
(b) Yes, those are absolute value bars.
(c) Yes, $|\cos(t)|$ is π-periodic.
(d) No, you don’t have to do integration by parts, but *tanstaafl* applies. Do you grok the unit stepping dance, the heavy side of thimble rigging??