General directions: *Read each problem carefully and do exactly what is requested. Full credit will be awarded only if you show all your work neatly, and it is correct. Write complete sentences, and use notation correctly. What is illegible or incomprehensible is worthless. Since the answer really consists of all the magic transformations, do not box your final result. Show me all the magic on the page. Communicate. Eschew obfuscation.*

1. (80 pts.) Solve each of the following differential equations or initial value problems. If there is no initial condition, obtain the general solution. [20 points/part]

(a) $\frac{dy}{dx} = 5x^4\cos^2(y)$; $y(0) = \frac{\pi}{3}$. The ODE is obviously separable.

There are infinitely many constant solutions to the ODE from the zeros of $\cos(y)$: $y(x) = (\pi/2)(2k+1)$, k any integer. None is a solution to the IVP. Separating variables and applying the initial conditions yields:

$$\tan(y) = x^5 + \sqrt{3} \ ,$$

an implicit solution, or better

$$y = \arctan(x^5 + \sqrt{3}) \ ,$$

and explicit solution alive and well on the whole real line.

(b) $\left(4e^{2x} + y^2\right)dx + \left(3y^2 + 2xy\right)dy = 0$

This varmint is plainly exact. A one-parameter family of solutions is given by

$$2e^{2x} + xy^2 + y^3 = C \ ,$$

where C is an arbitrary constant.

(c) $\frac{dy}{dx} + \frac{1}{x}y = \frac{2}{x^2+1}$; $y(1) = \ln(4)$.

The ODE of problem (c) is linear as written with an obvious integrating factor of

$$\mu(t) = e^{\ln(x)} = x \ \ for \ \ x > 0.$$

An explicit solution to the IVP is given by

$$y(x) = \frac{\ln(x^2+1) + \ln(2)}{x} \ \ for \ x > 0.$$

(d) $\left(x \sec\left(\frac{y}{x}\right) + y \right) dx - (x) dy = 0$; $y(1) = \frac{\pi}{6}$

The ODE is homogeneous. The degree of homogeneity is 1. Look at the "y/x" as a hint. Then write the equation in the form of $dy/dx = g(y/x)$ by doing suitable algebra carefully. After setting $y = vx$, substituting, and doing a bit more algebra, you will end up looking at the separable equation

$$\sec(v)dx - (x)dv = 0.$$

Separating variables and integrating leads you to

$$\int \cos(v)\, dv - \int \frac{1}{x}\, dx = C .$$

After doing that and integrating, and applying the initial condition you'll obtain

$$\sin\left(\frac{y}{x}\right) - \ln(x) = \frac{1}{2} \textit{ for } x > 0 ,$$

or an equivalent explicit solution like

$$y(x) = x \sin^{-1}\left(\ln(x) + \frac{1}{2}\right) \textit{ for } x > 0.$$

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2. (5 pts.) For certain values of the constant m the function $f(x) = x^m$ is a solution to the differential equation

$$x^2 y''(x) - 2x y'(x) = 0 .$$

Determine all such values of m.

It turns out that the function $f(x) = x^m$ is a solution to the differential equation precisely when

$$0 = -2x(mx^{m-1}) + x^2(m(m-1)x^{m-2} = \dots = (m(m-3))x^m$$

for all x. This is equivalent to $m = 0$ or $m = 3$.

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3. (5 pts.) It is known that every solution to the differential equation $y'' - y = 0$ is of the form

$$y(x) = c_1 e^x + c_2 e^{-x} .$$

Which of these functions satisfies the initial conditions $y(0) = 2$ and $y'(0) = 8$??

The initial conditions lead to the system of equations

$$\begin{cases} 2 = c_1 + c_2 \\ 8 = c_1 - c_2 \end{cases} \text{ which is equivalent to } \begin{cases} 5 = c_1 \\ -3 = c_2 \end{cases}$$

The solution to the IVP is given by

$$y(x) = 5e^x - 3e^{-x} .$$

4. (10 points) The following differential equation may be solved by either performing a substitution to reduce it to a separable equation or by performing a different substitution to reduce it to a homogeneous equation. Display the substitution to use and perform the reduction, **but do not attempt to solve the separable or homogeneous equation you obtain**.

$$(5x + 2y + 1)dx + (2x + y + 1)dy = 0$$

$$\begin{cases} 5h + 2k + 1 = 0 \\ 2h + k + 1 = 0 \end{cases} \text{ is equivalent to } \begin{cases} h = 1 \\ k = -3 \end{cases}$$

The substitution is $x = X + 1$ and $y = Y - 3$. The reduction results in the homogeneous DE

$$(5X + 2Y)dX + (2X + Y)dY = 0$$

Bonkers 10 Point Bonus: (a) The Fundamental Theorem of Calculus provides a neat formal solution involving a definite integral with respect to the variable t to the following dinky IVP:

$$y'(x) = e^{x^2} \text{ and } y(0) = 1.$$

What is that solution? (b) Unfortunately the function

$$g(x) = e^{x^2}$$

cannot be integrated in elementary terms. Use the answer to (a), the Maclaurin series for e^x, and term-by-term integration, to obtain a power series solution to the IVP. Write your answer using sigma notation. [Say where your work is! You don't have room here!]

(a)

$$y(x) = 1 + \int_0^x e^{t^2}\, dt \text{ for all } x.$$

(b)

$$\begin{aligned} y(x) &= 1 + \int_0^x e^{t^2}\, dt \\ &= 1 + \int_0^x \sum_{k=0}^{\infty} \frac{(t^2)^k}{k!}\, dt \\ &= 1 + \sum_{k=0}^{\infty} \int_0^x \frac{(t^2)^k}{k!}\, dt \\ &= 1 + \sum_{k=0}^{\infty} \int_0^x \frac{t^{2k}}{k!}\, dt \\ &= 1 + \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)k!} \text{ for all } x. \end{aligned}$$