General directions: *Show all essential work very neatly. Use correct notation when presenting your computations and arguments. Write using complete sentences. Be careful. Remember this: "=" denotes "equals" , "⇒" denotes "implies" , and "⇔" denotes "is equivalent to". Since the answer really consists of all the magic transformations, do not "box" your final results. Communicate. Show me all the magic on the page.*

1. (30 pts.) Obtain the general solution to each of the following linear homogeneous constant coefficient equations.

(a) $y''(x) - 3y'(x) - 10y(x) = 0$

Auxiliary Equation: $0 = m^2 - 3m - 10 = (m+2)(m-5)$

Roots of A.E.: $m = -2$ *or* $m = 5$

General Solution: $y = c_1 e^{-2x} + c_2 e^{5x}$

(b) $y''(x) - 6y'(x) + 10y(x) = 0$

Auxiliary Equation: $0 = m^2 - 6m + 10 = ((m-3)+i)((m-3)-i)$

Roots of A.E.: $m = 3+i$ *or* $m = 3-i$.

General Solution: $y = c_1 e^{3x}\cos(x) + c_2 e^{3x}\sin(x)$

(c) $\dfrac{d^5y}{dx^5} + 5\dfrac{d^4y}{dx^4} = 0$

Auxiliary Equation: $0 = m^5 + 5m^4 = m^4(m+5)$

Roots of A.E.: $m = 0$ *with multiplicity* 4, *or* $m = -5$

General Solution: $y = c_1 + c_2 x + c_3 x^2 + c_4 x^3 + c_5 e^{-5x}$

2. (10 pts.) Find the unique solution to the initial value problem

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} = 2e^x \;; \quad y(0) = 1, \quad y'(0) = -1,$$

given that a fundamental set of solutions to the corresponding homogeneous equation is { 1, e^x } and a particular integral to the original ODE is

$$y_p(x) = 2xe^x.$$

Hint: Save time. Use the stuff served on the platter with the cherry on top.
The general solution to the ODE is

$$y(x) = c_1 + c_2 e^x + 2xe^x.$$

By using the two initial conditions now, you can obtain an easy to solve linear system involving the two constants. Solving the system reveals that the solution to the initial value problem is

$$y(x) = 4 - 3e^x + 2xe^x.$$

3. (15 pts.) Find a particular integral, y_p, of the differential equation

$$y'' + y = 4\sec(x).$$

Obviously the driving function here is NOT a UC function. Thus, we must use variation of parameters to nab the culprit.

Corresponding Homogeneous: $y'' + y = 0$. F.S. = $\{\sin(x), \cos(x)\}$.

If $y_p = v_1\cos(x) + v_2\sin(x)$ then v_1' and v_2' are solutions to the following system:

$$\begin{cases} \cos(x)\, v_1' + \sin(x)\, v_2' = 0 \\ -\sin(x)\, v_1' + \cos(x)\, v_2' = 4\sec(x) \end{cases}$$

Solving the system yields $v_1' = -4\tan(x)$ and $v_2' = 4$. Thus, by integrating, we obtain

$$v_1 = -4\ln|\sec(x)| + c \;\; and \;\; v_2 = 4x + d.$$

Thus, a particular integral of the ODE above is

$$y_p = v_1\cos(x) + v_2\sin(x) = -4\ln|\sec(x)|\cos(x) + 4x\sin(x).$$

4. (10 pts.) Set up the correct linear combination of undetermined coefficient functions you would use to find a particular integral, y_p, of the O.D.E.

$$y'' - 4y' + 5y = x^2e^{-2x} + \sin(x)e^{2x}.$$

[**Warning:** *(a) If you skip a critical initial step, you will get no credit!! (b) Do not waste time attempting to find the numerical values of the coefficients!!*]

First, the corresponding homogeneous equation is

$$y'' - 4y' + 5y = 0.$$

which has an auxiliary equation given by $0 = m^2 - 4m + 5$.
Thus, $m = 2 + i$ or $m = 2 - i$, and a fundamental set of solutions for the corresponding homogeneous equation is $\{ \exp(2x)\cos(x), \exp(2x)\sin(x) \}$. Taking this into account, we may now write

$$y_p(x) = Ax^2e^{-2x} + Bxe^{-2x} + Ce^{-2x} + Dx\cos(x)e^{2x} + Ex\sin(x)e^{2x}$$

or something equivalent.

Silly 10 Point Bonus: Let $f(x) = x$ and $g(x) = \sin(x)$. (a) It is trivial to obtain a 4th order homogeneous linear constant coefficient ordinary differential equation with f and g as solutions. Do so.
(b) It's only slightly messier to obtain a 2nd order homogeneous linear ordinary differential equation with $\{ f, g \}$ as a fundamental set of solutions. Do so.
Hints: (a) Here one should expect that a fundamental set of solutions should be $\{ 1, x, \sin(x), \cos(x) \}$, arising from the auxiliary equation $m^2(m^2 + 1) = 0$. The constant coefficient equation has a family resemblance.
(b) Substitute f and g into the ODE

$$y'' + p(x)y' + q(x)y = 0$$

to obtain a linear system in $p(x)$ and $q(x)$. Find p and q and clean house.

5. (10 pts.) The factored auxiliary equation of a certain homogeneous linear O.D.E. with real constant coefficients is as follows:

$$m(m - 2\pi)^2(m - (5i))^3(m - (-5i))^3 = 0$$

(a) (5 pts.) Write down the general solution to the differential equation. [**WARNING: Be very careful. This will be graded Right or Wrong!!**]
(b) (5 pt.) What is the order of the differential equation?

$$y = c_1 + c_2 e^{2\pi x} + c_3 x e^{2\pi x} + c_4 \sin(5x) + c_5 \cos(5x) + c_6 x \sin(5x) + c_7 x \cos(5x) + c_8 x^2 \sin(5x) + c_9 x^2 \cos(5x)$$

The order of the differential equation is 9.

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6. (15 pts.) (a) Obtain the differential equation satisfied by the family of curves defined by the equation (*) below.
(b) Next, write down the differential equation that the orthogonal trajectories to the family of curves defined by (*) satisfy.
(c) Finally, solve the differential equation of part (b) to obtain the equation(s) defining the orthogonal trajectories. [These, after all, are another family of curves.]

$$(*) \quad x^2 = 2y - 1 + ce^{-2y} .$$

(a) Differentiating (*) with respect to x and then replacing c yields

$$2x = 2\frac{dy}{dx} - 2ce^{-2y}\frac{dy}{dx}$$

$$= \frac{dy}{dx}\left(2 - 2(x^2 - 2y + 1)e^{2y}e^{-2y}\right)$$

$$= \frac{dy}{dx}\left(4y - 2x^2\right).$$

Thus, a differential equation for the family of curves is given by

$$\frac{dy}{dx} = \frac{x}{2y - x^2}.$$

(b) An ODE for the orthogonal trajectories is now given by

$$\frac{dy}{dx} = \frac{x^2 - 2y}{x} ,$$

or equivalently,

$$\frac{dy}{dx} + \frac{2}{x}y = x. .$$

(c) This little first order linear equation has $\mu = x^2$ as an integrating factor. Using the standard recipe, a one-parameter family of solutions is given by

$$x^2 y = \frac{x^4}{4} + K .$$

Note: The Student Solutions’s Manual has some nonsense in its printed solution of this varmint.

7. (10 pts.) It turns out that the nonzero function f(x) = exp(x) is a solution to the homogeneous linear O.D.E.

$$(*) \qquad y''' - y = 0.$$

(a) Reduction of order with this solution involves making the substitution

$$y = ve^x$$

into equation (*) and then letting $w = v'$. Do this substitution and obtain the constant coefficient equation that w must satisfy. (b) Obtain a the general solution to the ODE that w satisfies and then stop.
(c) Explain very briefly why v can be obtained from w without actually integrating. ***Do not attempt to actually find v.***

(a) If we have

$$y = ve^x ,$$

then

$$y' = e^x v' + e^x v ,$$

$$y'' = e^x v'' + 2e^x v' + e^x v ,$$

and

$$y''' = e^x v''' + 3e^x v'' + 3e^x v' + e^x v .$$

Substituting y into (*) and then replacing v' using w implies that w must be a solution to

$$(**) \quad w'' + 3w' + 3w = 0.$$

(b) The auxiliary equation for (**) above is

$$m^2 + 3m + 3 = 0.$$

By using quadratic formula, it is easy to see that the roots are

$$m = -\frac{3}{2} \pm \frac{\sqrt{3}}{2} i$$

The general solution to (**), then, is

$$w = c_1 \cos(\frac{\sqrt{3}}{2} x) e^{-\frac{3}{2}x} + c_2 \sin(\frac{\sqrt{3}}{2} x) e^{-\frac{3}{2}x}.$$

(c) The equation $v' = w$ is a constant coefficient linear ODE with w being a UC function. [*Look at w above, Folks!*] Consequently, we can completely solve this ODE without performing any actual integrations.

Silly 10 Point Bonus: Let $f(x) = x$ and $g(x) = \sin(x)$. (a) It is trivial to obtain a 4th order homogeneous linear constant coefficient ordinary differential equation with f and g as solutions. Do so.
(b) It's only slightly messier to obtain a 2nd order homogeneous linear ordinary differential equation with { f , g } as a fundamental set of solutions. Do so. [Hints are on Page 2 of 4.]

(a) $y'''' + y'' = 0.$

(b) $(\sin(x) - x\cos(x))y'' - (x\sin(x))y' + (\sin(x))y = 0.$