Read Me First: Show all essential work very neatly. Use correct notation when presenting your computations and arguments. Write using complete sentences. Be careful. Remember this: "=" denotes "equals" , "⇒" denotes "implies" , and "⇔" denotes "is equivalent to". Do not "box" your answers. Communicate. Show me the all magic on the page.

Test #:

1. (10 pts.) Locate and classify the singular points of the following second order homogeneous O.D.E. Use complete sentences to describe the type of points and where they occur.

$$x^2(x-2)^2y'' + 2(x-2)y' + (x+1)y = 0$$

An equivalent equation in standard form is

$$y'' + \frac{2(x-2)}{x^2(x-2)^2}y' + \frac{x+1}{x^2(x-2)^2}y = 0.$$

From this, we can see easily that $x_0 = 0$ is an irregular singular point of the equation, and $x_0 = 2$ is a regular singular point. All other real numbers are ordinary points of the equation.

2. (15 pts.) Suppose

$$y(x) = \sum_{n=0}^{\infty} c_n x^n$$

is a solution of the homogeneous second order linear equation

$$y'' - y' + 2xy = 0.$$

Very neatly obtain the recurrence formula(s) needed to determine the coefficients of $y(x)$. DO NOT SPEND TIME ATTEMPTING TO GET ACTUAL COEFFICIENTS.

First,

$$\begin{aligned} 0 &= 2xy - y' + y'' \\ &= 2x\sum_{n=0}^{\infty} c_n x^n - \sum_{n=1}^{\infty} nc_n x^{n-1} + \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} \\ &= \sum_{n=1}^{\infty} 2c_{n-1}x^n - \sum_{n=0}^{\infty}(n+1)c_{n+1}x^n + \sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n \\ &= (2c_2 - c_1)x^0 + \sum_{n=1}^{\infty}\left[(n+2)(n+1)c_{n+2} - (n+1)c_{n+1} + 2c_{n-1}\right]x^n \end{aligned}$$

for all x near zero. From this you can deduce that $c_2 = c_1/2$, and that for $n \geq 1$, we have

$$c_{n+2} = \frac{(n+1)c_{n+1} - 2c_{n-1}}{(n+2)(n+1)}.$$

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3. (15 pts.) (a) If $f(t)$ and $g(t)$ are piecewise continuous functions defined for $t \geq 0$, what is the definition of the convolution of f with g, $(f*g)(t)$??

$$(f*g)(t) = \int_0^t f(x)g(t-x)\,dx$$

(b) Using only the definition of the convolution as a definite integral, not some fancy transform shenanigans, compute $(f*g)(t)$ when $f(t) = 3t^2$ and $g(t) = 4t$.

$$(f*g)(t) = \int_0^t f(x)g(t-x)\,dx = \int_0^t 3x^2 \cdot 4(t-x)\,dx$$
$$= \int_0^t 12tx^2 - 12x^3\,dx = \left(4tx^3 - 3x^4\right)\Big|_0^t = t^4.$$

(c) Using the Laplace transform table, compute the Laplace transform of $f*g$ when $f(t) = t\cdot\cos(t)$ and $g(t) = \exp(2t)$. [Do not attempt to simplify the algebra after computing the transform.]

$$\begin{aligned}\mathcal{L}\{(f*g)(t)\}(s) &= \mathcal{L}\{f(t)\}(s)\mathcal{L}\{g(t)\}(s)\\ &= \mathcal{L}\{t\cos(t)\}(s)\cdot\mathcal{L}\{e^{2t}\}(s)\\ &= \left[\frac{s^2-1}{(s^2+1)^2}\right]\cdot\left[\frac{1}{(s-2)}\right]\end{aligned}$$

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4. (10 pts.) (a) Suppose that $f(t)$ is defined for $t > 0$. What is the definition of the Laplace transform of f, $\mathcal{L}\{f(t)\}$, in terms of a definite integral??

$$\mathcal{L}\{f(t)\}(s) = \int_0^\infty f(t)e^{-st}\,dt = \lim_{R\to\infty}\int_0^R f(t)e^{-st}\,dt$$

for all s for which the integral converges.

(b) Using only the definition, not the table, compute the Laplace transform of

$$f(t) = \begin{cases} 0, & if\ \ 0 < t < 4 \\ 3, & if\ \ 4 < t. \end{cases}$$

$$\mathcal{L}\{f(t)\}(s) = \int_0^\infty f(t)e^{-st}\,dt = \lim_{R\to\infty}\left[\int_0^4 0e^{-st}\,dt + \int_4^R 3e^{-st}\,dt\right]$$
$$= \lim_{R\to\infty}\left[\frac{3e^{-4s}}{s} - \frac{3e^{-Rs}}{s}\right] = \frac{3e^{-4s}}{s} \quad provided\ s > 0.$$

Note: You may, of course, check your "answer" using #15 in the table.

5. (10 pts.) The equation below has a regular singular point at $x_0 = 0$.

$$x^2y'' - xy' + (x^2+1)y = 0$$

(a) Obtain the indicial equation for the ODE at $x_0 = 0$ and its two roots. (b) Then use all the information available and Theorem 6.3 to say what the two nontrivial linearly independent solutions given by theorem look like without attempting to obtain the coefficients of the power series involved.

To determine r, you need the indicial equation at $x_0 = 0$ and its roots. Now the indicial equation is $r(r-1) + p_0r + q_0 = 0$ where

$$p_0 = \lim_{x\to 0} x\left[\frac{-x}{x^2}\right] = -1 \text{ and } q_0 = \lim_{x\to 0} x^2\left[\frac{x^2+1}{x^2}\right] = 1.$$

Thus, the indicial equation is $r^2 - 2r + 1 = 0$, with a single root $r_1 = 1$ with multiplicity two. Consequently the two linearly independent solutions provided by Theorem 6.3 look like the following:

$$y_1(x) = |x|^1\sum_{n=0}^{\infty} c_nx^n \quad \text{and} \quad y_2(x) = |x|^2\sum_{n=0}^{\infty} d_nx^n + y_1(x)\ln|x|$$

6. (10 pts.) Compute $f(t) = \mathcal{L}^{-1}\{F(s)\}(t)$ when

(a) $$F(s) = \frac{7}{(2s+1)^3} \overset{(Work)}{=} \frac{7}{8(s-(-1/2))^3}$$

$$\mathcal{L}^{-1}\{F(s)\}(t) = \frac{7}{8}\left[\frac{1}{2}t^2e^{-(1/2)t}\right] = \frac{7}{16}t^2e^{-(1/2)t}$$

(b) $$F(s) = \frac{2s+12}{s^2+6s+13} \overset{(Work)}{=} \frac{2s+12}{(s+3)^2+4} = \frac{2(s+3)+6}{(s+3)^2+2^2}$$

$$\mathcal{L}^{-1}\{F(s)\}(t) = 2e^{-3t}\cos(2t) + 3e^{-3t}\sin(2t)$$

'Tis the usual prestidigitation of multiplication by '1' in the correct form or the addition of '0' suitably transmogrified. I move that we table the motion. Do I hear a second?

7. (5 pts.) Circle the letter corresponding to the correct response: If $F(s) = \mathcal{L}\{t^2\sin(bt)\}(s)$,then $F(s) =$

(a) $\left[\frac{2}{s^3}\right]\cdot\left[\frac{b}{(s^2+b^2)}\right]$

(b) $-\frac{d}{ds}\left[\frac{2bs}{(s^2+b^2)^2}\right]$

(c) $-\frac{d^2}{ds^2}\left[\frac{b}{s^2+b^2}\right]$

(d) $\frac{2bs}{s^2(s^2+b^2)^2}$

(e) None of the above.

Obviously (a) and (d) are utter nonsense since the transform of a product is NOT the product of the transforms. (c) is a near miss since the sign is wrong. It turns out that (b) provides the correct answer.

8. (15 pts.) Transform the given initial value problem into an algebraic equation in $\mathfrak{L}\{y\}$ and solve for $\mathfrak{L}\{y\}$. Do not take inverse transforms and do not attempt to combine terms over a common denominator. Be very careful.

$$3y''(t) - 5y'(t) + 7y(t) = \sin(2t) \; ; \; y(0) = 4 \, , \; y'(0) = 6 \; .$$

Applying the Laplace transform operator to BOTH SIDES OF THE ODE, using the two initial conditions, and then solving for the transform of y should reveal that

$$\mathfrak{L}\{y(t)\}(s) = \left[\frac{1}{3s^2 - 5s + 7}\right] \cdot \left[12s - 2 + \frac{2}{s^2+4}\right]$$

[A common error: Failure to parenthesize the first and second derivative's transform correctly.]

9. (10 pts.) The solution to a certain linear ordinary differential equation with coefficient functions that are analytic at $x_0 = 0$ is of the form

$$y(x) = \sum_{n=0}^{\infty} c_n x^n$$

where the coefficients satisfy the following equations:

$$-c_2 = 0, \quad c_0 + 3c_1 - 6c_3 = 0, \text{ and}$$

$$c_{n+2} = \frac{n(n+2)c_n + c_{n-1}}{(n+2)(n+1)} \quad \textit{for all } n \geq 2.$$

Determine the exact numerical value of the coefficients c_0, c_1, c_2, c_3, and c_4 for the particular solution that satisfies the initial conditions $y(0) = 0$ and $y'(0) = 1$.

$c_0 = y(0) = 0$

$c_1 = y'(0) = 1$

$c_2 = 0$

$c_3 = \dfrac{c_0 + 3c_1}{6} = \dfrac{1}{2}$

$c_4 = \dfrac{8c_2 + c_1}{(4)(3)} = \dfrac{1}{12}$

Silly 10 Point Bonus: Write the function

$$f(t) = \cos^5(bt)$$

as a linear combination of sine and/or cosines actually appearing in the Laplace transform table provided. Say where your work is, for it won't fit here.

From Euler's equation you should have readily that

$$\cos(bt) = \frac{1}{2}\left[e^{ibt} + e^{-ibt}\right] .$$

From the binomial theorem or Pascal's triangle or whatever, you should get

$$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5.$$

Thus, with much weeping and gnashing of algebraic teeth,

$$f(t) = \cos^5(bt) = \frac{1}{16}\cos(5bt) + \frac{5}{16}\cos(3bt) + \frac{10}{16}\cos(bt).$$