NAME:

Student Number: Exam Number:

Read Me First:

Read each problem carefully and do exactly what is requested. Full credit will be awarded only if you show all your work neatly, and it is correct. Use complete sentences and use notation correctly. Remember that what is illegible or incomprehensible is worthless. Communicate. Good Luck! [Total Points: 160]

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1. (80 pts.) Solve each of the following differential equations or initial value problems. If an initial condition is not given, display the general solution to the differential equation.
(20 pts./part)

(a) $y'' = y' + x$

(b) $\dfrac{d^2y}{dx^2} + y = \sec(x)\tan(x)$

1. (Continued) Solve each of the following differential equations or initial value problems. If an initial condition is not given, display the general solution to the differential equation. (20 pts./part)

(c) $(x+2y) + (2x+y)\dfrac{dy}{dx} = 0$

(d) $\dfrac{dy}{dx} = \tan^2(x)\sec(y)$; $y(0) = \dfrac{\pi}{6}$.

2. (10 pts.) Obtain the initial-value problem together with any additional equations that may be needed to solve each of the following problems. State clearly what your variables represent. *Do not waste time attempting to actually solve the differential equations or initial-value problems you provide.*

(a) // At 10 A.M. a person took a cup of hot instant coffee from a microwave oven in his kitchen and placed it on a nearby counter to cool. At that instant, the coffee was at 180° F, and ten minutes later it was 160° F. Assume the temperature in the kitchen was a constant 70° F. What was the temperature of the coffee at 10:15 A.M.? //

(b) // An eight pound weight is attached to the lower end of a coil spring suspended from the ceiling, and comes to rest in its equilibrium position, thereby stretching the spring 0.4 feet. The weight is then pulled down ~~up~~ 6 inches below its equilibrium position and released at t = 0. The resistance of the medium in pounds is numerically equal to 2x′, where x′ is the instantaneous velocity in feet per second. Assuming there are no externally impressed forces, determine the quasi period of the motion. //

3. (15 pts.) Without evaluating any integrals and using only the table provided, properties of the Laplace transform, and appropriate function identities, obtain the Laplace transform of each of the functions that follows. (5 pts./part)

(a) $h(t) = 4t\cos^2(3t) =$

$\mathfrak{L}\{h(t)\}(s) =$

(b)

$$f(t) = \begin{cases} 2, & \text{if } 0 < t < 1 \\ 4, & \text{if } 1 < t < 2 \\ 5, & \text{if } 2 < t < 3 \\ 0, & \text{if } 3 < t. \end{cases} =$$

$\mathfrak{L}\{f(t)\}(s) =$

(c)

$$g(t) = \begin{cases} 4t, & \text{if } 0 < t < 3 \\ 12, & \text{if } 3 < t. \end{cases} =$$

$\mathfrak{L}\{g(t)\}(s) =$

4. (10 pts.) (a) Obtain the recurrence formula(s) satisfied by the coefficients of the power series solution y at $x_0 = 0$, an ordinary point of the homogeneous ODE,

$$y'' - 2xy' = 0.$$

(b) Compute the first five (5) coefficients of the power series solution y_1 that satisfies the initial conditions $y(0) = 1$ and $y'(0) = -1$.

5. (10 pts.) The equation

$$x^2y'' + 5xy' + (x^2 + 4)y = 0$$

has a regular singular point at $x_0 = 0$. (a) Find the indicial equation of this O.D.E. at $x_0 = 0$ and determine its roots. (b) Then, using all the information now available and Theorem 6.3, say what the general solution at $x_0 = 0$ looks like without attempting to obtain the coefficients of the power series functions involved.

6. (15 pts.) Solve the following second order initial-value problem using only the Laplace transform machine.

$$y''(t) + 9y(t) = 3\sin(3t) \quad ; \; y(0) = 2, \; y'(0) = 3.$$

7. (10 pts.) Suppose that the Laplace transform of the solution to a certain initial value problem involving a linear differential equation with constant coefficients is given by

$$\mathfrak{L}\{y(t)\}(s) = \frac{e^{-2s} - 2e^{-5s}}{s^3}$$

Write the solution to the IVP in piecewise-defined form.

$y(t) =$

8. (10 pts.) (a) Given $f(x) = e^x$ is a nonzero solution to

$$(x-1)y'' - xy' + y = 0 ,$$

obtain a second, linearly independent solution by reduction of order.
(b) Use the Wronskian to prove the two solutions are linearly independent.

Bonkers 10 Point Bonus: You may attempt at most one of the following 2 bonus problems. Clearly indicate which one and where your work is.

(A) Suppose that the function $f(x) = x^n$, where n is a positive integer, is a solution to the homogeneous linear constant coefficient equation

(*) $$a_m y^{(m)} + a_{m-1} y^{(m-1)} + \dots + a_2 y'' + a_1 y' + a_0 y = 0$$

where $m > n$. What can you say about the coefficients of the ODE, and what can you say about its fundamental set of solutions?? Why???

(B) Suppose that $g(t)$ is continuous on the real line. Prove that

$$F(t) = \int_0^t g(x)\sin(t-x)\, dx$$

is a particular integral to the ODE

$$y''(t) + y(t) = g(t)$$

that satisfies the initial conditions

$$y(0) = 0 \quad and \quad y'(0) = 0$$

by direct computation of the needed derivatives and substitution into the appropriate equations.