1. (80 pts.) Solve each of the following differential equations or initial value problems. If there is no initial condition, obtain the general solution. [20 points/part]

(c) $y'(x) + y(x) = f(x)$ *and* $y(0) = -1$ where $f(x) = \begin{cases} 4 \ , & \text{if} \quad 0 \le x < 2 \\ 2x \ , & \text{if} \quad 2 \le x. \end{cases}$

Clearly the ODE is linear and has $\mu = e^x$ as an integrating factor. We really do not have to deal with this by handling a sequence of initial value problems. We may simply follow the usual algorithm for solving first order linear equations. Of course there is a catch. We will need to use the Fundamental Theorem of Calculus to build an antiderivative for a piece-wise defined function along the way. The neat thing is that we may do so and handle the initial condition simultaneously.

To begin, multiply the ODE by μ. Then we have

$$\frac{d(y(x)e^x)}{dx} = e^x f(x) \tag{*}$$

for each $x \ge 0$. Were we to follow the usual algorithm, we would end up looking at something like

$$y(x) = \left[\int e^x f(x)\ dx + C\right]e^{-x} \ .$$

This is unsatisfactory due to the presence of the mysterious antiderivative

$$\int e^x f(x)\ dx \ .$$

Who is this mysterious creature? And what is the appropriate value for the constant C ??

Plainly, the functions on each side of equation (*) above are continuous, and thus integrable. Consequently,

$$\int_0^x \frac{d(y(t)e^t)}{dt}\ dt = \int_0^x e^t f(t)\ dt$$

for each $x \ge 0$.

Since

$$\int_0^x \frac{d(y(t)e^t)}{dt}\ dt = y(x)e^x - y(0)e^0 = y(x)e^x - (-1) \ ,$$

we can obtain

$$y(x) = \left[\int_0^x e^t f(t)\ dt + (-1)\right]e^{-x} \ . \tag{**}$$

All that is left to do is compute the definite integral and then perform a little additional algebra.

Now observe that, since f above is piece-wise defined, it follows that the same is true for $f(t)e^t$. This means that the evaluation of the definite integral above will involve some casework.

For reference, here is

$$e^t f(t) = \begin{cases} 4e^t , & if \quad 0 \le t < 2 \\ 2t^t , & if \quad 2 \le t. \end{cases}$$

First, if $0 \le x < 2$, then

$$\int_0^x e^t f(t)\, dt = \int_0^x 4e^t\, dt = 4e^x - 4 \ .$$

Consequently, applying equation (**) from the preceeding page, it follows that

$$y(x) = \left[\int_0^x e^t f(t)\, dt + (-1) \right] e^{-x} = [\, 4e^x - 4 + (-1)\,]\, e^{-x} = 4 - 5e^{-x} \ .$$

Next, if $2 \le x$, then

$$\begin{aligned} \int_0^x e^t f(t)\, dt &= \int_0^2 e^t f(t)\, dt + \int_2^x e^t f(t)\, dt \\ &= \int_0^2 4e^t\, dt + \int_2^x 2te^t\, dt \\ &= 4e^2 - 4 + (\, 2xe^x - 2e^x\,) - (\, 4e^2 - 2e^2\,) \\ &= (2x - 2)e^x + 2e^2 - 4 \end{aligned} \ .$$

Again, applying equation (**) from the preceeding page, it follows that

$$\begin{aligned} y(x) &= \left[\int_0^x e^t f(t)\, dt + (-1) \right] e^{-x} \\ &= [\, (2x-2)e^x + 2e^2 - 4 + (-1)\,]\, e^{-x} \\ &= 2x - 2 + (\, 2e^2 - 5\,)\, e^{-x} \end{aligned} \ .$$

Thus, the solution to (c) is given by

$$y(x) = \begin{cases} 4 - 5e^{-x} & , \ 0 \le x < 2 \\ 2x - 2 + (2e^2 - 5)\, e^{-x} & , \ 2 \le x \end{cases} \ .$$

Question: Can we avoid all this casework?? Later in the course, we shall learn about the Laplace transform. It can ease some of the pain here, but not for free. After considering the above, you might now begin to appreciate how the gluing together process found in the original answer key might be *simpler* --- or not. e.t.