General directions: *Read each problem carefully and do exactly what is requested. Full credit will be awarded only if you show all your work neatly, and it is correct. Write complete sentences, and use notation correctly. Since the answer really consists of all the magic transformations, do not box your final result. Show me all the magic on the page. Communicate. Eschew obfuscation.*

1. (80 pts.) Solve each of the following differential equations or initial value problems. If there is no initial condition, obtain the general solution. [20 points/part]

(a) $\frac{dy}{dx} + 4xy = 8xy^{-3}$; $y(0) = 2$ This is obviously a Bernoulli equation. Since the DE is equivalent to the equation

$$y^3\frac{dy}{dx} + 4xy^4 = 8x ,$$

set $v = y^4$. Then $\frac{dv}{dx} = 4y^3\frac{dy}{dx}$. Multiplying by 4 and then substituting yields the linear equation

$$\frac{dv}{dx} + 16xv = 32x ,$$

which has $\mu = e^{\int 16x\,dx} = e^{8x^2}$ for all real x as an integrating factor. Multiplying the linear equation by μ yields

$$\frac{d[e^{8x^2}v]}{dx} = 32xe^{8x^2} .$$

Integrating both sides, replacing v, and determining the constant of integration using the initial condition, plus a little additional algebra, results in the implicit solution: $y^4 = 2 + 14e^{-8x^2}$. An explicit solution:

$$y(x) = (2 + 14e^{-8x^2})^{\frac{1}{4}}$$

(b) $\left(x\sec\left(\frac{y}{x}\right) + y\right)dx - x\,dy = 0$

This is a homogeneous equation. The degree of homogeneity is 1. Thus, write the equation in the form of $dy/dx = g(y/x)$ by doing suitable algebra carefully. After setting $y = vx$, substituting, and doing a bit more algebra, you will end up looking at the separable equation

$$\sec(v)\,dx - x\,dv = 0$$

Separating variables and integrating leads you to

$$\int \cos(v)\ dv - \int \frac{1}{x}\ dx = C .$$

After doing that and integrating, you'll obtain

$$\sin(v) - \ln|x| = C$$

or an equivalent beast. You may then finish this by replacing v above with y/x. If you are feeling feisty, get a family of explicit solutions.

(c) $y'(x) + y(x) = f(x)$ and $y(0) = -1$ where $f(x) = \begin{cases} 4, & \text{if } 0 \le x < 2 \\ 2x, & \text{if } 2 \le x. \end{cases}$

Clearly the ODE is linear and has $\mu = e^x$ as an integrating factor. We may cope with the piecewise defined function f by dealing with a sequence of initial value problems whose solutions, when glued together, will provide the solution to (c). [We shall leave many of the details to you.]

(I) $y' + y = 4$ and $y(0) = -1$: Multiplying the ODE by μ, doing the obligatory integration, and determining the constant of integration leads to the explicit solution, $y(x) = 4 - 5e^{-x}$ for x satisfying $0 \le x < 2$. [You really do need the explicit solution here to be able to deal effectively, easily with the initial condition of the next step.]

(II) $y' + y = 2x$ and $y(2) = \lim_{x\to 2^-} y(x) = \lim_{x\to 2^-} (4 - 5e^{-x}) = 4 - 5e^{-2}$ for $2 \le x$: Multiplying by μ leads us to $d(e^x y)/dx = 2xe^x$. Then integrating by parts and multiplying by e^{-x} yields $y(x) = 2x - 2 + de^{-x}$ for some number d. Using the initial condition, $y(2) = 4 - 5e^{-2}$, it turns out that $d = 2e^2 - 5$, so $y(x) = 2x - 2 + (2e^2-5)e^{-x}$ for $2 \le x$.

Thus, the solution to (c) is given by

$$y(x) = \begin{cases} 4 - 5e^{-x} & , 0 \le x < 2 \\ 2x - 2 + (2e^2-5)e^{-x} & , 2 \le x \end{cases} .$$

(d) $(e^{2x}y^2 - 2x)dx + (e^{2x}y)dy = 0$; $y(0) = 2$

Since $\frac{\partial}{\partial y}(e^{2x}y^2 - 2x) = 2ye^{2x} = \frac{\partial}{\partial x}(e^{2x}y)$, this equation is exact. Thus, there is a nice function $F(x,y)$ satisfying

$$(1)\quad \frac{\partial F}{\partial x} = e^{2x}y^2 - 2x \quad \text{and} \quad (2)\quad \frac{\partial F}{\partial y} = e^{2x}y .$$

It follows from (1) above that we have

$$(3)\quad F(x,y) = \int y^2e^{2x} - 2x\, dx = \frac{y^2}{2}e^{2x} - x^2 + c(y) ,$$

where $c(y)$ is some function of y whose identity we have yet to determine. Now using (2) and (3) together, $ye^{2x} = \frac{\partial}{\partial y}\left(\frac{y^2}{2}e^{xy} - x^2 + c(y)\right) = ye^{2x} + \frac{dc}{dy}(y)$,

which implies $\frac{dc}{dy}(y) = 0$. Integrating yields $c(y) = c_0$ for some constant c_0. Thus, $F(x,y) = \frac{y^2}{2}e^{2x} - x^2 + c_0$. A one-parameter family of implicit solutions is given by $\frac{y^2}{2}e^{2x} - x^2 = C$. Using the initial condition to determine C results in the implicit solution $\frac{y^2}{2}e^{2x} - x^2 = 2$.

2. (6 pts.) For what values of *m* is the function $f(x) = x^m$ a solution to the differential equation $4x^2y'' - 4xy' + 3y = 0$? // It turns out *f* above is a solution to the ODE above precisely when

$$0 = 4x^2 f'' - 4xf' + 3f = 3f - 4xf' + 4x^2 f''$$

$$= 3x^m - 4x(mx^{m-1}) + 4x^2(m(m-1)x^{m-2}) = (3 - 8m + 4m^2)x^m$$

for all *x*. It follows that *f* is a solution to the ODE if, and only if *m* is a root of the polynomial factor above. Finally,

$$0 = 4m^2 - 8m + 3 = (2m - 1)(2m - 3) \Leftrightarrow m = 1/2 \;\; or \;\; m = 3/2.$$

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3. (6 pts.) Every solution to the differential equation $y'' + 4y = 0$ is of the form $y(x) = c_1\sin(2x) + c_2\cos(2x)$. Which of these functions satisfies the initial conditions $y(0) = 8$ and $y'(0) = 2$??
// The initial conditions lead to the system of equations

$$\begin{cases} 8 = c_2 \\ 2 = 2c_1 \end{cases} \text{ which is equivalent to } \begin{cases} c_1 = 1 \\ c_2 = 8 \end{cases}$$

The solution to the IVP is given by

$$y(x) = \sin(2x) + 8\cos(2x) .$$

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4. (8 points) The following differential equation may be solved by either performing a substitution to reduce it to a separable equation or by performing a different substitution to reduce it to a homogeneous equation. Display the substitution to use and perform the reduction, **but do not attempt to solve the separable or homogeneous equation you obtain**.

$$(x - 2y + 1)dx + (4x - 3y - 6)dy = 0$$

The key to this puzzle is the solution to the linear system

$$\begin{cases} h - 2k + 1 = 0 \\ 4h - 3k - 6 = 0 \end{cases} \text{ which is equivalent to } \begin{cases} h = 3 \\ k = 2 \end{cases} .$$

Consequently, a suitable substitution is given by $x = X + 3$ and $y = Y + 2$. The substitution results in the homogeneous DE

$$(X - 2Y)dX + (4X - 3Y)dY = 0$$

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10 Point Bonus: (a) The *Fundamental Theorem of Calculus* provides a neat formal solution involving a definite integral with respect to the variable *t* to the following IVP:

$$y'(x) = \frac{1}{\sqrt{1+x^2}} \;\; and \;\; y(0) = 1.$$

Obtain that solution. (b) By evaluating the definite integral in (a), obtain an alias for the solution to the IVP. [Say where your work is below!]

(a) $y(x) = 1 + \int_0^x \frac{1}{\sqrt{1+t^2}}\,dt$ (b) $y(x) = 1 + \ln\left|\sqrt{1+x^2} + x\right|$

See Bonus Noise, Test1, Spring 2006, in the Tombs, for some details. e.t.