General directions: *Read each problem carefully and do exactly what is requested. Full credit will be awarded only if you show all your work neatly, and it is correct. Write complete sentences, and use notation correctly. Since the answer really consists of all the magic transformations, do not box your final result. Show me all the magic on the page. Communicate. Eschew obfuscation.*

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1. (10 pts.) Find the unique solution to the initial value problem

$$y'' - 2y' - 3y = 20\sin(x) \ ; \quad y(0) = 2, \quad y'(0) = 4,$$

given that a fundamental set of solutions to the corresponding homogeneous equation is $\{e^{3x}, e^{-x}\}$ and a particular integral to the original ODE is

$$y_p(x) = 2\cos(x) - 4\sin(x).$$

Hint: Save time. Use the stuff served on the platter with the cherry on top.
The general solution to the ODE is

$$y(x) = c_1 e^{3x} + c_2 e^{-x} + 2\cos(x) - 4\sin(x).$$

By using the two initial conditions now, you can obtain an easy to solve linear system involving the two constants. Solving the system reveals that the solution to the initial value problem is

$$y(x) = 2e^{3x} - 2e^{-x} + 2\cos(x) - 4\sin(x).$$

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2. (30 pts.) Obtain the general solution to each of the following linear homogeneous constant coefficient equations.

(a) $9y''(x) - 6y'(x) + y(x) = 0$

Auxiliary Equation: $0 = 9m^2 - 6m + 1 = (3m-1)^2$

Roots of A.E.: $m = 1/3$ *with multiplicity* 2

General Solution: $y = c_1 e^{\frac{x}{3}} + c_2 x e^{\frac{x}{3}}$

(b) $y''(x) - 2y'(x) + 10y(x) = 0$

Auxiliary Equation: $0 = m^2 - 2m + 10 = (m-1)^2 + 3^2$

Roots of A.E.: $m = 1 + 3i$ *or* $m = 1 - 3i$

General Solution: $y = c_1 e^x \cos(3x) + c_2 e^x \sin(3x)$

(c) $\dfrac{d^4y}{dx^4} + 2\dfrac{d^2y}{dx^2} + y = 0$

Auxiliary Equation: $0 = m^4 + 2m^2 + 1 = (m^2+1)^2 = (m+i)^2(x-i)^2$

Roots of A.E.: $m = i$, *or* $m = -i$, *each with multiplicity* 2

General Solution: $y = c_1\cos(x) + c_2\sin(x) + c_3 x\cos(x) + c_4 x\sin(x)$

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3. (15 pts.) Find a particular integral, y_p, of the differential equation

$$y'' + y = \sec^2(x).$$

Do not write the general solution.

Obviously the driving function here is NOT a UC function. Thus, we must use variation of parameters to nab the culprit.

Corresponding Homogeneous: $y'' + y = 0$. F.S. = {sin(x), cos(x)}.

If $y_p = v_1\cos(x) + v_2\sin(x)$, then v_1' and v_2' are solutions to the following system:

$$\begin{cases} \cos(x)\,v_1' + \sin(x)\,v_2' = 0 \\ -\sin(x)\,v_1' + \cos(x)\,v_2' = \sec^2(x) \end{cases}$$

Solving the system yields $v_1' = -\sec(x)\tan(x)$ and $v_2' = \sec(x)$. Thus, by integrating, we obtain

$$v_1 = -\sec(x) + c \quad \textit{and} \quad v_2 = \ln|\sec(x) + \tan(x)| + d.$$

Thus, a particular integral of the ODE above is

$$y_p = v_1\cos(x) + v_2\sin(x) = -\sec(x)\cos(x) + \ln|\sec(x) + \tan(x)|\sin(x)$$

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4. (10 pts.) Set up the correct linear combination of undetermined coefficient functions you would use to find a particular integral, y_p, of the O.D.E.

$$y'' + 4y' + 5y = 5xe^{-2x} + \sin(x)e^{-2x}.$$

[**Warning:** *(a) If you skip a critical initial step, you will get no credit!! (b) Do not waste time attempting to find the numerical values of the coefficients!!*]

First, the corresponding homogeneous equation is

$$y'' + 4y' + 5y = 0.$$

which has an auxiliary equation given by $0 = m^2 + 4m + 5$. Thus, $m = -2 + i$ or $m = -2 - i$. A fundamental set of solutions for the corresponding homogeneous equation is { exp(-2x)cos(x), exp(-2x)sin(x) }. Taking this into account, we may now write

$$y_p(x) = Ae^{-2x} + Bxe^{-2x} + Cx\cos(x)e^{-2x} + Dx\sin(x)e^{-2x}$$

or something equivalent.

Brief Silly 10 Point Bonus Answers:

(a) A fifth order constant coefficient homogeneous linear ODE with $f(x) = x^2$ and $g(x) = \sin(x)$ as solutions:

$$\frac{d^5y}{dx^5} + \frac{d^3y}{dx^3} = 0$$

(b) The key observation is that $f(x) = x^2$ cannot be a solution to any second order constant coefficient homogeneous linear equation on any interval on the real line. To see this, suppose that

$$ay'' + by' + cy = 0$$

is an equation with f above as a solution. Then by substituting, we see

$$(*) \quad 2a + 2bx + cx^2 = 0 .$$

If $a \neq 0$, (*) can only be true for at most two real numbers x.//

5. (10 pts.) The factored auxiliary equation of a certain homogeneous linear O.D.E. with real constant coefficients is as follows:

$$m(m-\pi)^2(m-3i)^2(m+3i)^2 = 0$$

(a) (5 pts.) Write down the general solution to the differential equation. [**WARNING: Be very careful. This will be graded Right or Wrong!!**]
(b) (5 pt.) What is the order of the differential equation?

$$y = c_1 + c_2 e^{\pi x} + c_3 x e^{\pi x}$$
$$+ c_4 \sin(3x) + c_5 \cos(3x)$$
$$+ c_6 x \sin(3x) + c_7 x \cos(3x)$$

The order of the differential equation is 7.

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6. (10 pts.) The nonzero function $f(x) = x + 1$ is a solution to the homogeneous linear O.D.E.

$$(*) \qquad (x+1)^2 y'' - 3(x+1)y' + 3y = 0.$$

(a) Reduction of order with this solution involves making the substitution

$$y = (x+1)v$$

into equation (*) and then letting $w = v'$. Do this substitution and obtain the first order linear homogeneous equation that w must satisfy.
(b) Finally, obtain an integrating factor, μ, for the first order linear ODE that w satisfies and then stop. ***Do not attempt to actually find w or v.***

(a) If we have

$$y = (x+1)v ,$$

then

$$y' = (x+1)v' + v \text{ , and } y'' = (x+1)v'' + 2v' .$$

Substituting y into (*), and then replacing v' using w implies that w must be a solution to

$$w' - \frac{1}{x+1} w = 0$$

after one cleans up the algebra a little.

(b) An integrating factor for the homogeneous linear equation that w satisfies is

$$\mu = \exp\left(\int \frac{-1}{x+1}\, dx \right) = \frac{1}{x+1}$$

since

$$\int \frac{-1}{x+1}\, dx = -\ln(x+1) + C = \ln\left(\frac{1}{x+1} \right) + C \quad \text{for } x > -1. \; .$$

7. (7 pts.) Obtain the initial-value problem needed to solve the following problem. State clearly what your variables represent. *Do not attempt to actually solve the differential equation or initial-value problem you provide*.

// A large 200 gallon tank initially contains 100 gallons of brine in which there is dissolved 10 pounds of salt. At time $t_0 = 0$, brine containing 4 pounds of dissolved salt per gallon flows into the tank at the rate of 5 gallons/minute. The mixture is kept uniform by stirring, and the well-stirred mixture flows out of the tank at the slower rate of 4 gallons/minute. How much salt is in the tank at the moment that it overflows. //

Let $x(t)$ denote the amount of salt, in lbs., in the tank at time t, in minutes. Since the amount of water in the tank at time t is $100 + (5 - 4)t$ gallons, the IVP that $x(t)$ satisfies is

$$x'(t) = 20 - \frac{4x(t)}{100+t} \;\; ; \;\; x(0) = 10 \; .$$

8. (8 pts.) Obtain the general solution to the following ODE:

$$y'' = x + y \; .$$

The ODE above is obviously equivalent to

(*) $$y'' - y = x \; .$$

This is linear with constant coefficients.
The corresponding homogeneous equation,

$$y'' - y = 0$$

has the trivial auxiliary equation

$$(m-1)(m+1) = m^2 - 1 = 0 \; .$$

Consequently, a fundamental set of solutions to this equation consists of

$$F.S. = \{e^x, e^{-x}\} \; .$$

The driving function of (*) is so simple a U.C. function that you should be able to see that a particulat integral is given by

$$y_p = -x$$

The general solution: $y(x) = c_1 e^x + c_2 e^{-x} - x$

Silly 10 Point Bonus: Let $f(x) = x^2$ and $g(x) = \sin(x)$. (a) It is trivial to obtain a 5th order homogeneous linear constant coefficient ordinary differential equation with real-valued coefficients with f and g as solutions. Do so.
(b) Show there is no 2nd order homogeneous linear constant coefficient ODE with $\{ f , g \}$ as a fundamental set of solutions with real number coefficients. [Say where your work is, for it won't fit here.]

Brief answers may be found on the bottom of Page 2 of 4.