Read Me First: Communicate. Show all essential work very neatly and use correct notation when presenting your computations and arguments. Write using complete sentences. Show me all the magic on the page. Eschew obfuscation.

Test #:

1. (10 pts.) Locate and classify the singular points of the following second order homogeneous O.D.E. Use complete sentences to describe the type of points and where they occur.

$$(x^2-2x)^2y'' + (x-2)y' + (x+1)y = 0$$

An equivalent equation in standard form is

$$y'' + \frac{x-2}{(x(x-2))^2}y' + \frac{x+1}{(x(x-2))^2}y = 0.$$

From this, we can see easily that $x_0 = 0$ is an irregular singular point of the equation, and $x_0 = 2$ is a regular singular point. All other real numbers are ordinary points of the equation.

2. (15 pts.) Suppose

$$y(x) = \sum_{n=0}^{\infty} c_n x^n$$

is a solution of the homogeneous second order linear equation

$$y'' - 4y' + x^2 y = 0.$$

Very neatly obtain the recurrence formula(s) needed to determine the coefficients of $y(x)$. *DO NOT WASTE TIME ATTEMPTING TO GET THE NUMERICAL VALUES OF THE COEFFICIENTS.*

First,

$$\begin{aligned} 0 &= x^2y - 4y' + y'' \\ &= x^2\sum_{n=0}^{\infty} c_n x^n - \sum_{n=1}^{\infty} 4nc_n x^{n-1} + \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} \\ &= \sum_{n=2}^{\infty} c_{n-2}x^n - \sum_{n=0}^{\infty} 4(n+1)c_{n+1}x^n + \sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n \\ &= (2c_2-4c_1)x^0 + (6c_3-8c_2)x^1 + \sum_{n=2}^{\infty}[(n+2)(n+1)c_{n+2} - 4(n+1)c_{n+1} + c_{n-2}]x^n \end{aligned}$$

for all x near zero. From this you can deduce that we have

$$c_2 = 2c_1,$$

$$c_3 = \frac{4c_2}{3}, \text{ and}$$

$$c_{n+2} = \frac{4(n+1)c_{n+1} - c_{n-2}}{(n+2)(n+1)} \quad \text{for } n \geq 2.$$

3. (15 pts.) (a) If $f(t)$ and $g(t)$ are piece-wise continuous functions defined for $t \geq 0$, what is the definition of the convolution of f with g, $(f*g)(t)$??

$$(f*g)(t) = \int_0^t f(x)\,g(t-x)\,dx$$

(b) Suppose $f(t) = 3t$ and $g(t) = 2e^{-t}$.

Using only the definition of the convolution as a definite integral, not some fancy transform shenanigans, compute $(f*g)(t)$.

$$(f*g)(t) = \int_0^t f(x)\,g(t-x)\,dx = \int_0^t 3x \cdot 2e^{-(t-x)}\,dx$$

$$= 6e^{-t}\int_0^t xe^x\,dx = 6e^{-t}\left[(xe^x - e^x)\Big|_0^t\right] = \ldots = 6t - 6 + 6e^{-t}$$

(c) Suppose $h(t) = (f*g)(t)$, where $f(t) = e^{2t}$ and $g(t) = e^{-2t}\cos(3t)$.

Using the table, compute the Laplace transform of h.

$$\mathfrak{L}\{h(t)\}(s) = \mathfrak{L}\{(f*g)(t)\}(s) = \mathfrak{L}\{f(t)\}(s)\,\mathfrak{L}\{g(t)\}(s)$$

$$= \mathfrak{L}\{e^{2t}\}(s)\cdot\mathfrak{L}\{e^{-2t}\cos(3t)\}(s)$$

$$= \left[\frac{1}{s-2}\right]\cdot\left[\frac{s+2}{(s+2)^2+9}\right]$$

4. (10 pts.) The equation below has a regular singular point at $x_0 = 0$.

$$x^2y'' - 6xy' + (x^2-8)y = 0$$

(a) Obtain the indicial equation for the ODE at $x_0 = 0$ and its two roots.
(b) Then use all the information available and Theorem 6.3 to say what the two non-trivial linearly independent solutions given by the theorem look like without attempting to obtain the coefficients of the power series.

// To determine r, you need the indicial equation at $x_0 = 0$ and its roots. Now the indicial equation is $r(r-1) + p_0 r + q_0 = 0$ where

$$p_0 = \lim_{x\to 0} x\left[\frac{-6x}{x^2}\right] = -6 \text{ and } q_0 = \lim_{x\to 0} x^2\left[\frac{x^2-8}{x^2}\right] = -8.$$

Thus, the indicial equation is $r^2 - 7r - 8 = 0$, with two roots $r_1 = 8$ and $r_2 = -1$. Consequently the two linearly independent solutions provided by Theorem 6.3 look like the following:

$$y_1(x) = |x|^8\sum_{n=0}^{\infty} c_n x^n \quad \text{and} \quad y_2(x) = |x|^{-1}\sum_{n=0}^{\infty} d_n x^n + C\,y_1(x)\ln|x|$$

5. (10 pts.) (a) Suppose that $f(t)$ is defined for $t \geq 0$. What is the definition of the Laplace transform of f in terms of a definite integral??

$$\mathfrak{L}\{f(t)\}(s) = \int_0^\infty f(t)e^{-st}\,dt = \lim_{R\to\infty}\int_0^R f(t)e^{-st}\,dt$$

for all s for which the integral converges.

(b) Using only the definition, not the table, compute the Laplace transform of

$$f(t) = \begin{cases} 0 & , \text{ if } 0 < t < 1 \\ 2e^t & , \text{ if } 1 < t. \end{cases}$$

$$\mathfrak{L}\{f(t)\}(s) = \int_0^\infty f(t)e^{-st}\,dt = \lim_{R\to\infty}\left[\int_0^1 0e^{-st}\,dt + \int_1^R 2e^t e^{-st}\,dt\right]$$

$$= \lim_{R\to\infty}\left[\frac{2e^{-(s-1)}}{s-1} - \frac{2e^{-R(s-1)}}{s-1}\right] = \frac{2e^{-(s-1)}}{s-1} \quad \textit{provided } s > 1.$$

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6. (10 pts.) Compute $f(t) = \mathfrak{L}^{-1}\{F(s)\}(t)$ when

(a) $F(s) = \dfrac{5s}{s^2-6s+9} \overset{(Work)}{=} \dfrac{5}{s-3} + \dfrac{15}{(s-3)^2}$ *using a pfd.*

$$f(t) = \mathfrak{L}^{-1}\{F(s)\}(t) = 5e^{3t} + 15te^{3t}$$

(b) $F(s) = \dfrac{2s+12}{s^2+6s+13} \overset{(Work)}{=} \dfrac{2(s+3) + (3)(2)}{(s+3)^2 + 2^2}$

$$f(t) = \mathfrak{L}^{-1}\{F(s)\}(t) = 2e^{-3t}\cos(2t) + 3e^{-3t}\sin(2t)$$

'Tis just the usual magic of multiplication by '1' in the correct form or the addition of '0' suitably transmogrified with linearity tossed into the mix.

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7. (5 pts.) The following initial-value problem may be converted to an equivalent initial-value problem where the ODE has constant coefficients. Convert the problem and stop. Don't attempt to solve the transformed IVP.

$$x^2y'' - 5xy' + 8y = 2x^3; \quad y(2) = 0, \quad y'(2) = -8, \quad (x > 0).$$

The ODE is an Euler-Cauchy equation. By letting $x = e^t$, and $w(t) = y(e^t)$, so that $y(x) = w(\ln(x))$ for $x > 0$, the ODE above transforms into the following ODE in $w(t)$:

$$w''(t) - 6w'(t) + 8w(t) = 2e^{3t}.$$

The transformed initial conditions are as follows:

$$w(\ln(2)) = y(e^{\ln(2)}) = y(2) = 0 \quad \textit{and} \quad w'(\ln(2)) = e^{\ln(2)}y'(e^{\ln(2)}) = -16.$$

[This gets really really really confusing if you overload the dependent variable, as in classical notation!!!]

8. (15 pts.) Using only the Laplace transform machine, solve the following first order initial-value problem.

$$y'(t) - y(t) = t^3 e^t \; ; \; y(0) = 3 \; .$$

By taking the Laplace Transform of both sides of the differential equation, and using the initial condition, we have

$$\mathcal{L}\{y'(t)\}(s) - \mathcal{L}\{y(t)\}(s) = \mathcal{L}\{t^3 e^t\}$$

$$\Rightarrow \quad s\mathcal{L}\{y(t)\}(s) - y(0) - \mathcal{L}\{y(t)\}(s) = \frac{3!}{(s-1)^4}$$

$$\Rightarrow \quad (s-1)\mathcal{L}\{y(t)\}(s) = 3 + \frac{3!}{(s-1)^4}$$

$$\Rightarrow \quad \mathcal{L}\{y(t)\}(s) = \frac{3}{s-1} + \frac{3!}{(s-1)^5}.$$

By taking inverse transforms now, we quickly obtain

$$y(t) = 3e^t + \frac{1}{4}t^4 e^t.$$

9. (10 pts.) The solution to a certain linear ordinary differential equation with coefficient functions analytic at $x_0 = 0$ is of the form

$$y(x) = \sum_{n=0}^{\infty} c_n x^n$$

where the coefficients satisfy the following equations:

$$c_2 = \frac{c_1}{2},$$

$$c_3 = \frac{c_2}{3}, \; and$$

$$c_{n+2} = \frac{(n+1)c_{n+1} - c_{n-2}}{(n+2)(n+1)} \quad for \; n \geq 2.$$

Determine the exact numerical value of the coefficients c_0, c_1, c_2, c_3, and c_4 for the particular solution that satisfies the initial conditions $y(0) = -1$ and $y'(0) = 2$.

$c_0 = y(0) = -1$ $\qquad$ $c_1 = y'(0) = 2$

$c_2 = \frac{c_1}{2} = 1$ $\qquad$ $c_3 = \frac{c_2}{3} = \frac{1}{3}$

$$c_4 = \frac{3c_3 - c_0}{(4)(3)} = \frac{2}{12} = \frac{1}{6}$$

Silly 10 Point Bonus: Suppose g is continuous and f is continuously differentiable for $t \geq 0$. By using the Laplace transform, obtain a formula for the derivative of

$$F(t) = \int_0^t g(x)f(t-x)\,dx \; .$$

[To actually prove the formula is valid requires additional work. Say where your work is below.] *This may be seen in de-t3-bo.wp.*