
Silly 10 Point Bonus: Suppose g is continuous and f is continuously differentiable for $t \geq 0$. By using the Laplace transform, obtain a formula for the derivative of

$$F(t) = \int_0^t g(x) f(t-x)\, dx .$$

[To actually prove the formula is valid requires additional work. Say where your work is below.]

The key here is to recognize that F is the convolution of g with f. To use the Laplace transform, we shall assume that g and f are nice enough. Then

$$\begin{aligned}
\mathfrak{L}\{F'(t)\}(s) &= s\mathfrak{L}\{(F(t)\}(s) - F(0) \\
&= s\mathfrak{L}\{F(t)\}(s) \\
&= [\mathfrak{L}\{g(t)\}(s)][s\mathfrak{L}\{f(t)\}(s)] \\
&= [\mathfrak{L}\{g(t)\}(s)][\mathfrak{L}\{f'(t)\}(s) + f(0)] \\
&= \mathfrak{L}\{(g*f')(t)\}(s) + f(0)\mathfrak{L}\{g(t)\}(s).
\end{aligned}$$

By considering inverse transforms and the definition of convolution, it follows that

$$F'(t) = \int_0^t g(x) f'(t-x)\, dx + f(0)g(t) .$$

It turns out that this is important since using the Laplace transform frequently results in solutions given in terms of convolutions. Of course the formula is valid even for functions not having a Laplace transform. A proof, in generality, is quite similar in spirit to the standard proof of the interesting part of the Fundamental Theorem of Calculus.