**Read Me First**: Communicate. Show all essential work very neatly and use correct notation when presenting your computations and arguments. Write using complete sentences. Show me the all magic on the page. Eschew obfuscation.

Test #:

1. (10 pts.) Locate and classify the singular points of the following second order homogeneous O.D.E. Use complete sentences to describe the type of points and where they occur.

$$(x^2-2x)^2 y'' + (x-2)y' + (x+1)y = 0$$

2. (15 pts.) Suppose

$$y(x) = \sum_{n=0}^{\infty} c_n x^n$$

is a solution of the homogeneous second order linear equation

$$y'' - 4y' + x^2 y = 0.$$

Very neatly obtain the recurrence formula(s) needed to determine the coefficients of $y(x)$. *DO NOT WASTE TIME ATTEMPTING TO GET THE NUMERICAL VALUES OF THE COEFFICIENTS.*

3. (15 pts.) (a) If $f(t)$ and $g(t)$ are piece-wise continuous functions defined for $t \geq 0$, what is the definition of the convolution of $f$ with $g$, $(f*g)(t)$??

$(f*g)(t) =$

(b) Suppose $f(t) = 3t$ and $g(t) = 2e^{-t}$ .

Using only the definition of the convolution as a definite integral, not some fancy transform shenanigans, compute $(f*g)(t)$.

$(f*g)(t) =$

(c) Suppose $h(t) = (f*g)(t)$, where $f(t) = e^{2t}$ and $g(t) = e^{-2t}\cos(3t)$ .

Using the table, compute the Laplace transform of $h$.

$\mathfrak{L}\{h(t)\}(s) =$

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4. (10 pts.) The equation below has a regular singular point at $x_0 = 0$.

$$x^2y'' - 6xy' + (x^2-8)y = 0$$

(a) Obtain the indicial equation for the ODE at $x_0 = 0$ and its two roots.
(b) Then use all the information available and Theorem 6.3 to say what the two non-trivial linearly independent solutions given by the theorem look like without attempting to obtain the coefficients of the power series.

5. (10 pts.) (a) Suppose that $f(t)$ is defined for $t \geq 0$. What is the definition of the Laplace transform of $f$ in terms of a definite integral??

$\mathfrak{L}\{f(t)\}(s) =$

(b) Using only the definition, not the table, compute the Laplace transform of

$$f(t) = \begin{cases} 0 & , \ if \ \ 0 < t < 1 \\ 2e^{t} & , \ if \ \ 1 < t. \end{cases}$$

$\mathfrak{L}\{f(t)\}(s) =$

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6. (10 pts.) Compute $f(t) = \mathfrak{L}^{-1}\{F(s)\}(t)$ when

(a) $F(s) = \dfrac{5s}{s^2-6s+9}$

(b) $F(s) = \dfrac{2s+12}{s^2+6s+13}$

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7. (5 pts.) The following initial-value problem may be converted to an equivalent initial-value problem where the ODE has constant coefficients. Convert the problem and stop. Don't attempt to solve the transformed IVP.

$$x^2y'' - 5xy' + 8y = 2x^3; \quad y(2) = 0, \quad y'(2) = -8, \quad (x > 0).$$

8. (15 pts.) Using only the Laplace transform machine, solve the following first order initial-value problem.

$$y'(t) - y(t) = t^3 e^t \; ; \; y(0) = 3 \; .$$

9. (10 pts.) The solution to a certain linear ordinary differential equation with coefficient functions analytic at $x_0 = 0$ is of the form

$$y(x) = \sum_{n=0}^{\infty} c_n x^n$$

where the coefficients satisfy the following equations:

$$c_2 = \frac{c_1}{2},$$

$$c_3 = \frac{c_2}{3}, \text{ and}$$

$$c_{n+2} = \frac{(n+1)\,c_{n+1} - c_{n-2}}{(n+2)(n+1)} \quad \text{for } n \geq 2 .$$

Determine the exact numerical value of the coefficients $c_0$, $c_1$, $c_2$, $c_3$, and $c_4$ for the particular solution that satisfies the initial conditions $y(0) = -1$ and $y'(0) = 2$.

$c_0$ = $c_1$ =

$c_2$ = $c_3$ =

$c_4$ =

**Silly 10 Point Bonus:** Suppose $g$ is continuous and $f$ is continuously differentiable for $t \geq 0$. By using the Laplace transform, obtain a formula for the derivative of

$$F(t) = \int_0^t g(x)\,f(t-x)\;dx \; .$$

[To actually prove the formula is valid requires additional work. Say where your work is below.]