General directions: Read each problem carefully and do exactly what is requested. Show all your work neatly. Use complete sentences and use notation correctly. Make your arguments and proofs as complete as possible. Remember that what is illegible or incomprehensible is worthless. Show me all the magic on the page.

1. (10 pts.) Construct the truth tables for each of the following compound propositions in the space provided:

(a) $\neg p \to q$

p	$\neg p$	q	$\neg p \to q$
T	F	T	T
T	F	F	T
F	T	T	T
F	T	F	F

(b) $\neg(p \to q)$

p	q	$p \to q$	$\neg(p \to q)$
T	T	T	F
T	F	F	T
F	T	T	F
F	F	T	F

2. (5 pts.) Suppose p and q are propositions. What is the difference between "$p \leftrightarrow q$" and "$p \Leftrightarrow q$"?

"$p \leftrightarrow q$" is the biconditional which has truth value "true" precisely when either both p and q are "true" or both p and q are "false". "$p \Leftrightarrow q$" denotes the assertion that p and q are logically equivalent, which means that $p \leftrightarrow q$ is a tautology, i.e., a proposition that is always true. Observe that $p \leftrightarrow q$ being a tautology necessitates both p and q must have the same truth value. This doesn't say that both p and q are true!!

3. (10 pts.) Write the contrapositive and converse of the statement, "If horses have wings, then the Pegasus needs dramamine," and label these unambiguously. Which of these fails to be equivalent to the original statement?

Contrapositive : "If Pegasus doesn't need dramamine, then horses don't have wings."

Converse : "If Pegasus needs dramamine, then horses have wings."

The converse fails to be equivalent to the original proposition generally.

4. (15 pts.) Let F(x,y) be the statement "x can fool y". The universe of discourse is all people. Use quantifiers to express each of the following statements:

(a) There is someone whom Frodo can fool.

$$(\exists x)F(\text{Frodo},x)$$

(b) Gandalf fools everyone.

$$(\forall x)F(\text{Gandalf},x)$$

(c) The only person Frodo can fool is himself.

There are several equivalent formulations for this gem. All require appropriate quantification.

$$(\forall x)(\ F(\text{Frodo},x) \rightarrow x = \text{Frodo}\ )$$

$$(\forall x)(\ x \neq \text{Frodo} \rightarrow \neg F(\text{Frodo},x))$$

$$\neg(\exists x)(\ (x \neq \text{Frodo}) \wedge F(\text{Frodo},x)\ )$$

Note that all of these are logically equivalent. There are evidently other equivalent answers. The last may be the most obvious. Translated into English, it is something like this: "There is no one different from Frodo whom Frodo fools." Note, too, this is different from what one would get when symbolizing, "Frodo fools only himself."

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5. (5 pts.) Determine the truth value of each of the following statements if the universe of discourse of each variable is the set of natural numbers, $\mathbb{N} = \{0,1,2,...\}$.

(a) $(\forall x)(\exists y)(x + y = y)$ F or False

(b) $(\exists x)(\forall y)(x + y = y)$ T or True

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6. (5 pts.) If A is a countable set and B is an uncountable set, must A - B be countable? Briefly explain.

Yes, A - B must be countable because $A - B \subseteq A$, a countable set. [Subsets of countable sets must be countable.]

7. (15 pts.) Suppose $A = \{\emptyset,\{\emptyset\},2\}$ and $B = \{2,3\}$. Then

$A \cup B = \{\emptyset, \{\emptyset\}, 2, 3\}$

$A \times B = \{(\emptyset,2), (\emptyset,3), (\{\emptyset\},2), (\{\emptyset\},3), (2,2), (2,3)\}$

$|P(A)| = 2^{|A|} = 2^3 = 8$

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8. (5 pts.) Suppose A and B are subsets of a universal set U. Show that $A - B = \emptyset \rightarrow A \subseteq B$.
[Hints: (1) $x \in A \rightarrow x \notin \emptyset \rightarrow \ldots$.
(2) $A - B = \emptyset \rightarrow \forall x(x \in A - B \rightarrow x \in \emptyset)$. The contrapositive of the implication within the parentheses here is useful in dealing with the ellipsis in hint #1.]

Element-wise Proof using the hints:

We must show $\forall x(x \in A \rightarrow x \in B)$ when $A - B = \emptyset$. To this end, let $x \in A$ be arbitrary. From Hint #1, $x \in A \rightarrow x \notin \emptyset$. From the hypothesis, $A - B = \emptyset$, and using Hint #2, it follows that we have $x \notin \emptyset \rightarrow x \notin A - B$. Thus, $x \in A$ and $x \in A \rightarrow x \notin A - B$. Consequently, $x \notin A - B$. From De Morgan's Law,

$$x \notin A - B \Leftrightarrow \neg(x \in A) \vee (x \in B) \Leftrightarrow (x \in A \rightarrow x \in B).$$

Thus we have $x \in A$ and $(x \in A \rightarrow x \in B) \rightarrow x \in B$. Since x was arbitrary, we are finished.

An Alternative Proof using some logical equivalences:

$$\begin{aligned}
A - B = \emptyset &\Leftrightarrow \forall x(x \in A - B \rightarrow x \in \emptyset) \\
&\Leftrightarrow \forall x(\neg(x \in \emptyset) \rightarrow \neg(x \in A - B)) \\
&\Leftrightarrow \forall x((x \in \emptyset) \vee (\neg(x \in A) \vee (x \in B))) && \text{[DeMor.]} \\
&\Leftrightarrow \forall x(\neg(x \in A) \vee (x \in B)) && \text{[Ident.]} \\
&\Leftrightarrow \forall x(x \in A \rightarrow x \in B) \\
&\Leftrightarrow A \subseteq B
\end{aligned}$$

Since $A - B = \emptyset \Leftrightarrow A \subseteq B$, $A - B = \emptyset \leftrightarrow A \subseteq B$ is a tautology. Thus, $A - B = \emptyset \rightarrow A \subseteq B$ follows.

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9. (5 pts.) If $f:X \rightarrow Y$ is a function, f^{-1} may be used to denote two quite different things. What are they? [Use complete sentences.](a) If $A \subseteq Y$, then $f^{-1}(A)$ denotes the inverse image of A, i.e.,$f^{-1}(A) = \{ x \in X : f(x) \in A \}$. (b) If f is one-to-one and onto, i.e., a bijection or one-to-one correspondence, f^{-1} may be used to denote the inverse function of f. Here, of course, $f^{-1}: Y \rightarrow X$ is defined by $f^{-1}(a) = b$ if, and only if $f(b) = a$ for each $a \in Y$.

10. (15 pts.) Suppose that $f:\mathbb{R} \to Z$ is the function defined by the formula $f(x)= \lfloor x \rfloor$, and suppose that $A = \{x \varepsilon \mathbb{R} | -3 \le x \le 3\}$ and $B = \{x \varepsilon \mathbb{R} | -1 < x \le \pi\}$. Using appropriate notation, give each of the following.

$A - B = \{ x : x \varepsilon \mathbb{R} \text{ and } -3 \le x \le -1 \} = [-3,-1]$

$f(B) = \{ -1, 0, 1, 2, 3 \}$

$f^{-1}(\{1,3\}) = f^{-1}(\{1\}) \cup f^{-1}(\{3\})$

$= [1,2) \cup [3,4)$

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11. (5 pts.) What is the value of the following sum of terms of a geometric progression? [Hint: You may wish to re-index the varmint.]

$$\sum_{j=1}^{8} 2^j = \sum_{k=0}^{7} 2^{k+1} = 2\cdot\sum_{k=0}^{7} 2^k = 2[2^8 - 1]/[2-1] = 2^9-2$$

$= 510$

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12. (5 pts.) Suppose $g:A \to B$ and $f:B \to C$ are functions. Prove exactly one of the following propositions. Indicate clearly which you are demonstrating.

(a) If $f\circ g:A \to C$ is injective, then g is injective.

Proof: We must show

$(\forall a_1)(\forall a_2)(a_1 \varepsilon A \wedge a_2 \varepsilon A \wedge g(a_1) = g(a_2) \to a_1 = a_2)$.

To this end, suppose $a_1 \varepsilon A \wedge a_2 \varepsilon A \wedge g(a_1) = g(a_2)$ for arbitrary a_1 and a_2. Since f is a function, from the definition of composition, we have $f\circ g(a_1) = f(g(a_1)=f(g(a_2)) = f\circ g(a_2)$. Finally $f\circ g$ injective yields this: $f\circ g(a_1) = f\circ g(a_2) \to a_1 = a_2$. Thus, $a_1 \varepsilon A \wedge a_2 \varepsilon A \wedge g(a_1) = g(a_2) \to a_1 = a_2$ for arbitrary a_1 and a_2, and we are finished.

(b) If $f\circ g:A \to C$ is surjective, then f is surjective.

Proof: We must show

$(\forall c)(c \varepsilon C \to (\exists b)(b \varepsilon B \wedge f(b) = c))$.

To deal with this, pretend that $c \varepsilon C$ is arbitrary. $f\circ g$ surjective $\to (\exists a)(a \varepsilon A \wedge f\circ g(a) = c)$. Let $a_0 \varepsilon A$ be such an element with $f\circ g(a_0) = c$. Let $b_0 = g(a_0)$. Since g is a function, $b_0 \varepsilon B$. From the definition of composition, we have $f(b_0) = f(g(a_0)) = f\circ g(a_0) = c$. Consequently, we have shown $c \varepsilon C \to (\exists b)(b \varepsilon B \wedge f(b) = c)$. Since $c \varepsilon C$ was arbitrary, we are finished.