
General directions: Read each problem carefully and do exactly what is requested. Show all your work neatly. Use complete sentences and use notation correctly. Make your arguments and proofs as complete as possible. Remember that what is illegible or incomprehensible is worthless. **Test Number:**

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1. (15 pts.) Suppose matrices A, B, and C are as given below.

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & -2 & 8 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -7 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} -1 & -2 \\ -2 & 1 \\ -3 & -2 \\ -4 & 1 \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} 4 & 1 \\ 3 & -2 \\ 2 & 1 \\ 1 & -2 \end{bmatrix}$$

Then

$$\mathbf{B} + \mathbf{C} = \begin{bmatrix} 3 & -1 \\ 1 & -1 \\ -1 & -1 \\ -3 & -1 \end{bmatrix}$$

$$AC = \begin{bmatrix} 8 & -17 \\ 0 & 0 \\ -5 & 15 \end{bmatrix}$$

$$\mathbf{B}^t = \begin{bmatrix} -1 & -2 & -3 & -4 \\ -2 & 1 & -2 & 1 \end{bmatrix}$$

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2. (10 pts.) Truth or ... . Label each of the following assertions with "true" or "false". **Be sure to write out the entire word.**

(a) The number of one-to-one functions from a set with 15 elements to a set with 8 elements is P(15,8). **False**
Answer:____________________

(b) The number of one-to-one functions from a set with 8 elements to a set with 15 elements is P(15,8). **True**
Answer:____________________

(c) The coefficient of x^3y^5 in the expansion of $(x + y)^8$ is the number of 5 element subsets of an 8 element set. **True**
Answer:____________________

(d) The coefficient of x^3y^5 in the expansion of $(x + y)^8$ is the number of 3-permutations of an 8 element set. **False**
Answer:____________________

(e) The total number of ways to assign truth values to five true-false problems with at least 2 of the answers being true is given by

$$\sum_{k=2}^{5} C(5,k).$$

True
Answer:____________________

3. (25 pts.) A standard deck of playing cards consists of 52 cards divided into 4 suits of 13 cards each. There are 2 black suits, spades and clubs, and there are 2 red suits, hearts and diamonds. Each suit consists of 10 spot cards ranging from 1 or ace through 10, and 3 face cards: jack, queen, and king. A hand in draw poker consists of 5 cards.

For each of the following questions, either provide an appropriate integer **or** indicate unambiguously how your answer would be computed **or** explain why the given assertion is true. [Correct notation involving combinations or permutations may be used.]

(a) How many draw poker hands are there?

n = C(52,5) or equivalent.

(b) How many ways can Frodo arrange the cards in his draw poker hand??

n = P(5,5) = 5! or equivalent.

(c) How many draw poker hands have exactly 3 cards of one kind, e.g., 3 kings, or 3 aces, or 3 fives, and two other cards that are different from the three that are alike??

n = $13 \cdot C(4,3) \cdot C(48,2)$ or equivalent.

(d) Every draw poker hand has at least two cards of the same suit. Why? What magic principle applies here??

Since there are 5 cards in a hand and only 4 suits, the magical pigeonhole principle implies that there are at least 2 cards having the same suit.

(e) Suppose the deck is well shuffled, so we may safely assume that the cards are randomly ordered. How many cards must Frodo be dealt from the top of the deck to ensure that he has 7 cards from the same suit.

n = 25 does the trick. This is an easy application of the generalized pigeonhole principle. From the generalized pigeonhole principle, we need to find the smallest positive integer n so that $\lceil n/4 \rceil = 7$. Thus, n = ((7 - 1)*4) + 1 = 25 will do.

4. (15 pts.) (a) Suppose f is a function defined recursively by f(0) = 1 and f(n+1) = -2f(n) for n = 0,1,2,3,.... Then

f(1) = -2f(0) = -2
f(2) = -2f(1) = 4
f(3) = -2f(2) = -8
f(4) = -2f(3) = 16

(b) Give a recursive definition for the sequence $\{a_n\}$, where n = 1,2,3,..., if $a_n = 6n$. [Be very careful with the quantification used in the induction step.]

Basis Step: $a_1 = 6$

Induction Step: $(\forall n)(n \in \mathbb{N}^+ \rightarrow a_{n+1} = a_n + 6)$

or equivalent, say something like

$$(\forall n \in \mathbb{N}^+)(a_{n+1} = a_n + 6).$$

Note: The set defining the universe of discourse must be part of the induction step here.

(c) Give a recursive definition of the set S of positive integers that have a remainder of 2 upon division by 3. Be very careful with the quantification in the induction step.

Basis Step: $2 \in S$

Induction Step: $(\forall x)(x \in S \rightarrow x + 3 \in S)$ or equivalent.

Note: $(\forall x)(\forall y)((x \in S$ and $y \in S) \rightarrow x + y \in S)$ will not work here. Why?? 2 + 2 = 4 has a remainder of 1 upon division by 3, not the required 2.//

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6. (10 pts.) Use mathematical induction to prove that a set with n elements has n(n-1)/2 subsets containing exactly two elements whenever n is a positive integer greater than or equal to 2.// Formally, let P(n) denote the propositional function "A set with n elements has n(n-1)/2 subsets containing exactly two elements." We shall use induction to verify the truth of P(n) for positive integers $n \geq 2$. [P(n) is actually true for $n \geq 1$.!!]
Basis Step: Observe that a set with 2 elements has exactly 1 subset with 2 elements, and $1 = (2 \cdot (2 - 1))/2$. Thus, P(2) is true.
Inductive Step: Suppose n is an arbitrary integer that is greater than or equal to 2, and pretend that P(n) is true. Let S be an arbitrary set with n + 1 elements. Pick an element x_0 from S. The set $S - \{x_0\}$ has n elements. The two element subsets of S either contain x_0, or they do not. The two element subsets of S that do not contain x_0 are two element subsets of $S - \{x_0\}$. From the induction hypothesis, there are n(n-1)/2 of these. The two element subsets of S that do contain x_0, may be built by taking x_0 and one element from $S - \{x_0\}$. So there are n 2 element subsets of S that contain x_0. In total there are $[n(n-1)/2] + n = n(n+1)/2 = ((n+1) \cdot ((n+1)-1))/2$ subsets of S that have 2 elements. Since, for arbitrary $n \geq 2$, we have verified $P(n) \rightarrow P(n+1)$, after applying U.G., we have satisfied both the hypotheses for induction.
An application of modus ponens completes the proof.//

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7. (10 pts.) Using a complete sentences, correctly name the fallacy exemplified by each of the following invalid arguments.

(a) "If x is any real number with $x > 3$, then $x^2 > 9$. Suppose that $x \leq 3$. Then $x^2 \leq 9$."

This is the infamous fallacy of denying the hypothesis. [Some logic texts call this the fallacy of denying the antecedent.]

(b) "If x is any real number with $x > 3$, then $x^2 > 9$. Suppose that $x^2 > 9$. Then $x > 3$."

This is the infamous fallacy of affirming the conclusion. [Some logic texts call this the fallacy of affirming the consequent.]

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8. (15 pts.) This is a valid argument: "All dogs are carnivorous. Some animals are dogs. Therefore some animals are carnivorous." The validity of the argument can be seen easily by symbolizing the argument using propositional functions and quantifiers as follows:

Define propositional functions as follows:

D(x) : "x is a dog." C(x) : "x is carnivorous."
A(x) : "x is an animal."

Then the argument translates into this:

$(\forall x)(D(x) \rightarrow C(x))$
$(\exists x)(A(x) \wedge D(x))$

$\therefore\ (\exists x)(A(x) \wedge C(x))$

In the proof of validity which follows provide the missing justification(s) for each step. You should explicitly cite rules of inference and quantification, hypotheses, and preceding steps by number. [Note: The lower case latin letters 'a','b','c' denote individuals in the universe of discourse.]

1.	$(\forall x)(D(x) \rightarrow C(x))$	Justification:	Hypothesis
2.	$(\exists x)(A(x) \wedge D(x))$	Justification:	Hypothesis
3.	$A(c) \wedge D(c)$	Justification:	2, Existential Instantiation
4.	$D(c) \rightarrow C(c)$	Justification:	1, Universal Instantiation
5.	$D(c) \wedge A(c)$	Justification:	3, $\wedge$ is commutative.
6.	$D(c)$	Justification:	5, Simplification
7.	$C(c)$	Justification:	4, 6, Modus Ponens
8.	$A(c)$	Justification:	3, Simplification
9.	$A(c) \wedge C(c)$	Justification:	8, 7, Conjunction
10.	$(\exists x)(A(x) \wedge C(x))$	Justification:	9, Existential Generalization