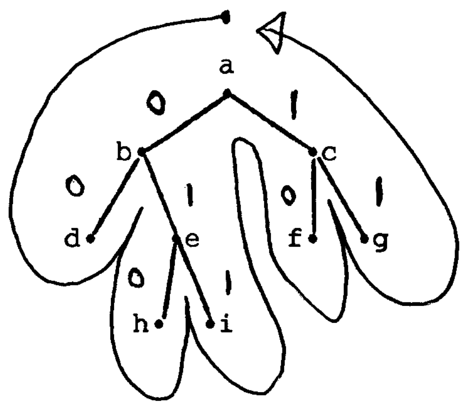
General directions: Read each problem carefully and do exactly what is requested. Show all your work neatly. Use complete sentences and use notation correctly. Make your arguments and proofs as complete as possible. Remember that what is illegible or incomprehensible is worthless.

1. (15 pts.) Given the ordered rooted tree below, list the vertices in the order in which they are visited in (a) a preorder traversal of the tree, (b) an inorder traversal of the tree, and (c) a postorder traversal of the tree. Place your lists in the appropriate place below.



(a) preorder traversal : a b d e h i c f g

(b) inorder traversal : d b h e i a f c g

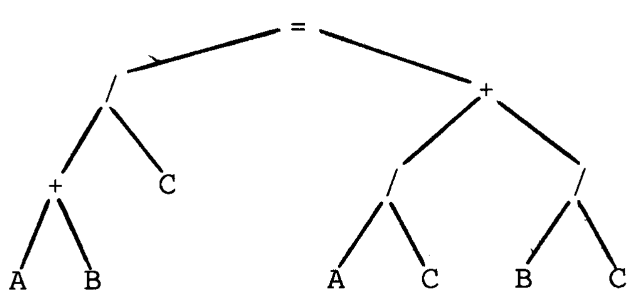
(c) postorder traversal : d h i e b f g c a

2. (10 pts.) (a) Using the rooted binary tree in Problem 1 to obtain appropriate prefix codes, produce the string of zeros and ones that encodes the word **dig**.

dig : 0001111

After labelling the edges as above, simply follow the paths from root to the leaves. Then the required encodings are as follows: d : 00, i : 011, and g : 11.

(b) Construct the ordered rooted binary tree representing the following algebraic identity: ((A + B)/C) = ((A/C) + (B/C))



3. (15 pts.) Suppose R and S are relations on the set A = {a,b,c} represented by the matrices given below.

$$\mathbf{M}_R = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{M}_S = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

(a) If we assume that the matrices were constructed using the order listed above for the set A, then

S = { (a,a), (a,c), (b,b), (c,a), (c,c) }

(b) What is the matrix representing $R \cup S$?

$$M_{R \cup S} = \quad \mathbf{M}_R \vee \mathbf{M}_R \quad = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

(c) Construct the matrix representing the relation $T = R^{-1}$ that is the inverse of R.

$$\mathbf{M}_T = \quad \mathbf{M}_R^T \quad = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

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4. (10 pts.) (a) What is the numerical value of the postfix expression below?

3 2 * 2 ↑ 5 3 - 8 4 / * - = 6 2 ↑ 5 3 - 8 4 / * -

= 36 5 3 - 8 4 / * -

= 36 2 8 4 / * -

= 36 2 2 * -

= 36 4 -

= 32

(b) What is the numerical value of the prefix expression below?

+ - * 2 3 5 / ↑ 2 3 4 = + - * 2 3 5 / 8 4

= + - * 2 3 5 2

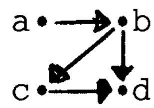
= + - 6 5 2

= + 1 2

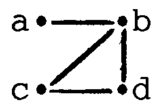
= 3

5. (15 pts.) (a) Draw a directed graph G_1 whose adjacency matrix is given on the left below.// V_1 = { a, b, c, d }.

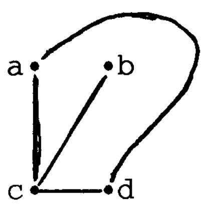
$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$



(b) Now draw the underlying undirected graph G_2 for the directed graph G_1 of part (a) of this problem.// V_2 = { a, b, c, d }.



(c) Is $G_2 = (V_2,E_2)$, above, isomorphic to the simple graph $G_3 = (V_3,E_3)$ given below? Either display an isomorphism $f:V_2 \to V_3$ or very briefly explain why there is no such function by revealing an invariant that one graph has that the other doesn't.



Yes, define $f:V_2 \to V_3$ by f(a) = b, f(b) = c, f(c) = a, and f(d) = d. Then

{ a ,b } ε E_2 $\leftrightarrow$ { f(a), f(b) } = { b, c } ε E_3,
{ b ,c } ε E_2 $\leftrightarrow$ { f(b), f(c) } = { c, a } ε E_3,
{ b ,d } ε E_2 $\leftrightarrow$ { f(b), f(d) } = { c, d } ε E_3,
{ c ,d } ε E_2 $\leftrightarrow$ { f(c), f(d) } = { a, d } ε E_3.

[You could also use f(a) = b, f(b) = c, f(c) = d, and f(d) = a.]

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6. (10 pts.) Recall that the composition of two relations R and S on a set A is given by

$$S\circ R = \{ (a,c) \ \varepsilon\ A \times A \mid (\exists b)(b \ \varepsilon\ A \text{ and } (a,b)\ \varepsilon\ R \text{ and } (b,c)\ \varepsilon\ S)\}.$$

Also, recall that the n^{th} composition power of a relation on a set is defined recursively by $R^1 = R$, and for each n ε $\mathbb{N}$, if $n \geq 1$, then $R^{n+1} = R^n\circ R$.

Prove that if R is a reflexive relation on a nonempty set A, then $R \subseteq R^n$ for every positive integer n. // Remark: Evidently we need to do a proof by induction since the n^{th} composition power of a relation is defined recursively.

Proof. The basis step is true since $R^1 = R$ implies $R \subseteq R^1$ for any relation R on a nonempty set A. To deal with the induction step, it suffices to show that if the relation R is reflexive, then $(\forall n \varepsilon \mathbb{N}^+)(R \subseteq R^n \to R \subseteq R^{n+1})$. To this end, pretend n is an arbitrary positive integer, R is reflexive, and $R \subseteq R^n$. Let (a,b) ε R be arbitrary. Then (a,b) ε R and the induction hypothesis, $R \subseteq R^n$, implies (a,b) ε R^n. R reflexive and a ε A $\to$ (a,a) ε R. The definition of the composition $R^n\circ R = R^{n+1}$, (a,a) ε R, and (a,b) ε R^n implies (a,b) ε R^{n+1}. Thus we have shown that $R \subseteq R^n \to R \subseteq R^{n+1}$ for an arbitrary $n\varepsilon\mathbb{N}^+$. So we have verified -- ugh -- that $(\forall n\varepsilon\mathbb{N}^+)(R \subseteq R^n \to R \subseteq R^{n+1})$. Since we have verified the hypotheses of the induction axiom, modus ponens wraps things up: If R is reflexive, $(\forall n\varepsilon\mathbb{N}^+)(R \subseteq R^n)$.//

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7. (15 pts.) (a) How many vertices does a tree with 37 edges have?

$$|V| = |E| + 1 = 38$$

(b) What is the maximum number of leaves that a binary tree of height 6 can have?

If l denotes the number of leaves, then $l \le 2^6 = 64$. The maximum occurs when the tree is complete, i.e., all the leaves are at the same level.

(c) If a full 3-ary tree has 24 internal vertices, how many leaves does it have?

A full 3-ary tree with 24 internal vertices must have $|V| = (3)\cdot(24) + 1 = 73$ vertices. Since 24 are internal, there must be $l = 73 - 24 = 49$ leaves.

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8. (10 pts.) Suppose that A is the set consisting of all real-valued functions with domain consisting of the interval [-1,1] . Let R be the relation on the set A defined as follows:

$$R = \{ (f,g) \mid (\exists C)(C \in \mathbb{R} \text{ and } f(0) - g(0) = C) \}$$

Prove that R is an equivalence relation on the set A.

Proof. We must establish that R is reflexive, symmetric, and transitive to show that it is an equivalence relation.

To see that R is reflexive, suppose that f is an element of the set A. Then $f:[-1,1] \to \mathbb{R}$ is a function. Since we have that $f(0) - f(0) = 0$ and $0 \in \mathbb{R}$, $(f,f) \in R$. Since f was arbitrary, it follows that $(\forall f)(f \in A \to (f,f) \in R)$. Thus R is reflexive.

To show that R is symmetric, pretend that f and g are members of A with $(f,g) \in R$. It follows from the definition of the relation that there is some real number C for which $f(0) - g(0) = C$. Now $C \in \mathbb{R}$ implies $-C \in \mathbb{R}$. Since we have $g(0) - f(0) = -C$, it follows that $(g,f) \in R$. Consequently, we have shown that R is symmetric.

Finally, to verify that R is transitive, let f, g, and h be elements of A with $(f,g) \in R$ and $(g,h) \in R$. It follows from the definition of the relation R that there are real numbers, say C_1 and C_2, such that $f(0) - g(0) = C_1$ and $g(0) - h(0) = C_2$. Since $C_1 + C_2 \in \mathbb{R}$ when C_1 and C_2 are in $\mathbb{R}$, and we have that $f(0) - h(0) = f(0) - g(0) + g(0) - h(0) = C_1 + C_2$, $(f,h) \in R$. Consequently, R is transitive.//

Question: How many equivalence classes does this relation have? One, for any two real-valued functions on [-1,1] are equivalent under this relation. Why??

TIMTOWTDI: The "induction step" in Problem 6 may be handled somewhat differently. If $(a,b) \in R$, then R reflexive implies that $(b,b) \in R$. The induction hypothesis, $R \subseteq R^n$, may then be used to imply that $(b,b) \in R^n$. Then the definition of the composition may be applied to infer that $(a,b) \in R^n \circ R = R^{n+1}$. Yadda, yadda, yadda....