Chapter 13

Gravitation

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Goals for Chapter 13

- To calculate the gravitational forces that bodies exert on each other
- To relate weight to the gravitational force
- To use the generalized expression for gravitational potential energy
- To study the characteristics of circular orbits
- To investigate the laws governing planetary motion
- To look at the characteristics of black holes

Newton's law of gravitation

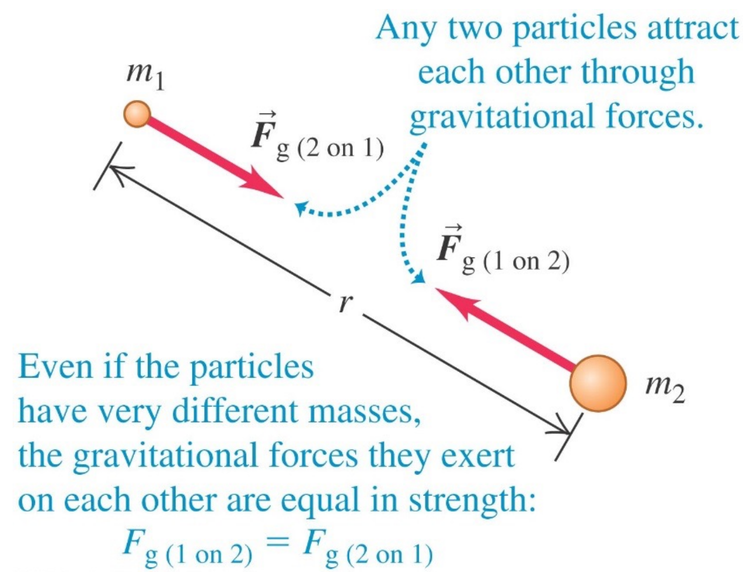
- *Law of gravitation*: Every particle of matter attracts every other particle with a force that is directly proportional to the product of their masses and inversely proportional to the square of the distance between them.

- The gravitational force can be expressed mathematically as

$$\boxed{F_g = Gm_1m_2/r^2,}$$

where G is the *gravitational constant.*

$$G = 6.67428(67)\times10^{-11}\ \mathrm{N{\cdot}m^2/kg^2}$$



$m_1 = 100\ \mathrm{kg}$ $m_2 = 100\ \mathrm{kg}$

$r = 1\ \mathrm{m}$

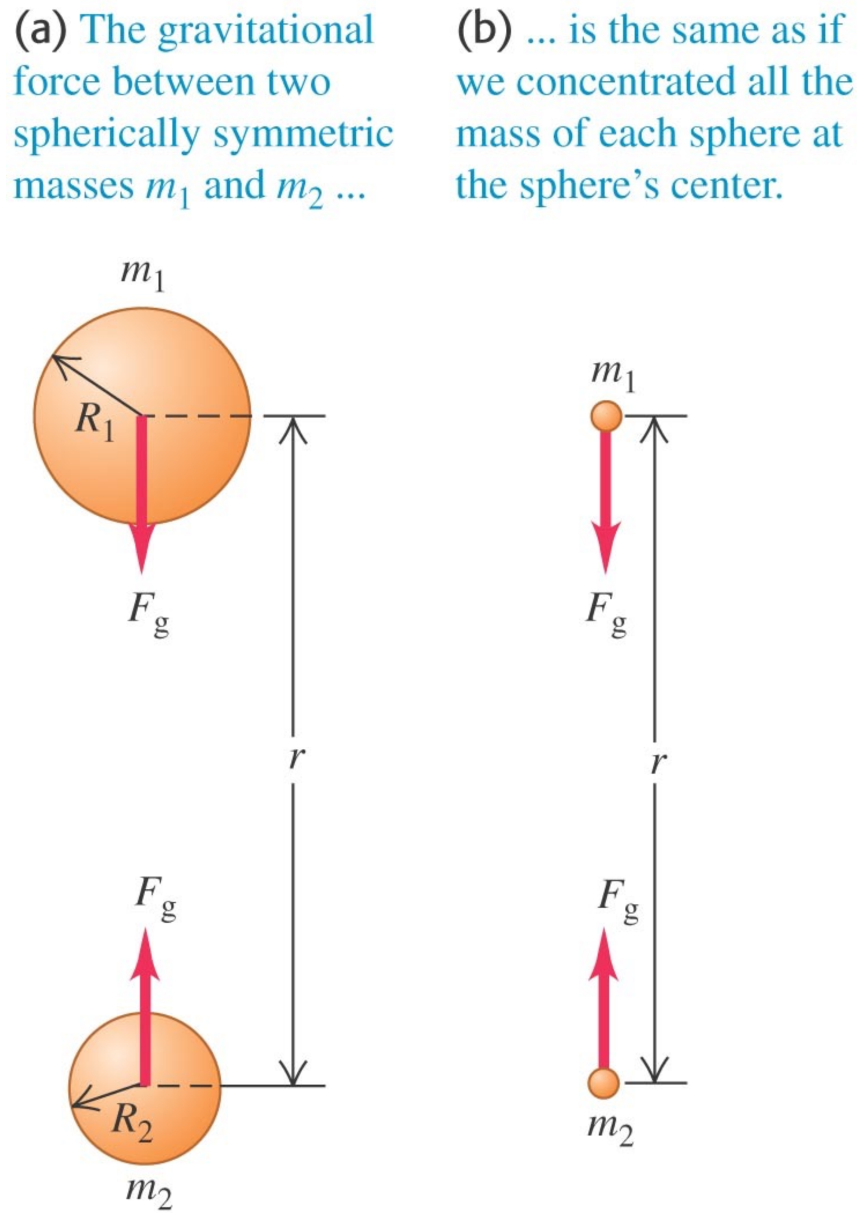
$$F = \frac{6.67\times10^{-11}\ \mathrm{N}\cdot\cancel{\mathrm{m}^2}/\cancel{\mathrm{kg}^2}\times(100\ \cancel{\mathrm{kg}})^2}{(1\ \cancel{\mathrm{m}})^2}$$

$$= 6.67\times10^{-11}\times10^{4}\ \mathrm{N}$$

$$= 6.67\times10^{-7}\ \mathrm{N}$$

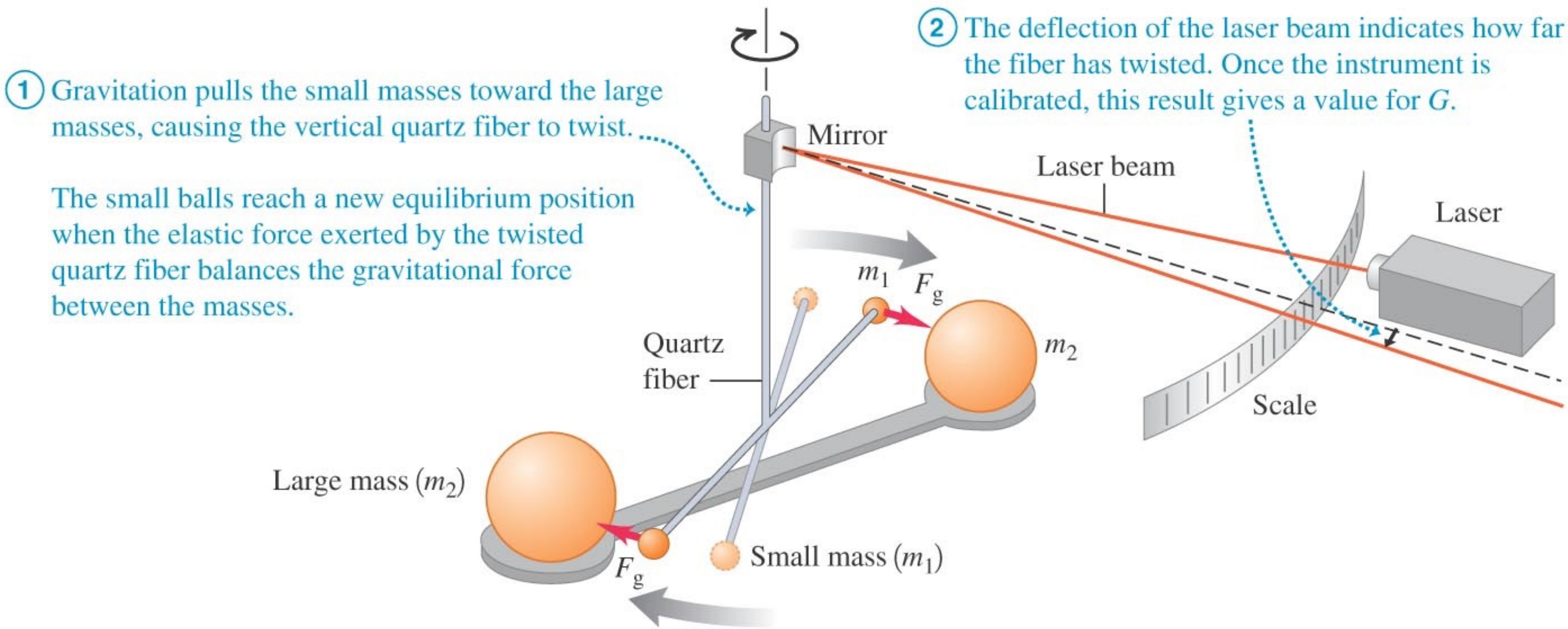
Gravitation and spherically symmetric bodies

- The gravitational interaction of bodies having *spherically symmetric* mass distributions is the same as if all their mass were concentrated at their centers.



Determining the value of *G*

- In 1798 Henry Cavendish made the first measurement of the value of *G*. Figure below illustrates his method.



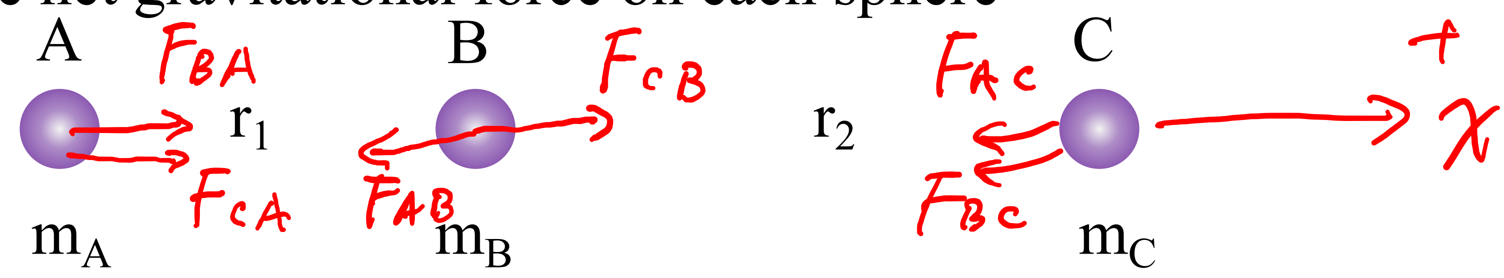
Example: gravitational calculations

B D

A C

Example

- Find the net gravitational force on each sphere



A: $F_A = F_{BA} + F_{CA} = G\frac{m_A m_B}{r_1^2} + G\frac{m_A m_C}{(r_1+r_2)^2}$, right

B: $F_B = F_{CB} - F_{AB} = G\frac{m_B m_C}{r_2^2} - G\frac{m_B m_A}{r_1^2}$, + given by the sign.

C: $F_C = F_{AC} + F_{BC} = G\frac{m_C m_A}{(r_1+r_2)^2} + G\frac{m_C m_B}{r_2^2}$

or: $F_C = -G\frac{m_C m_A}{(r_1+r_2)^2} - G\frac{m_C m_B}{r_2^2}$, − left.

Weight

- The *weight* of a body is the total gravitational force exerted on it by all other bodies in the universe.

- At the surface of the earth, we can neglect all other gravitational forces, so a body's weight is

$$w = Gm_E m/R_E^2.$$

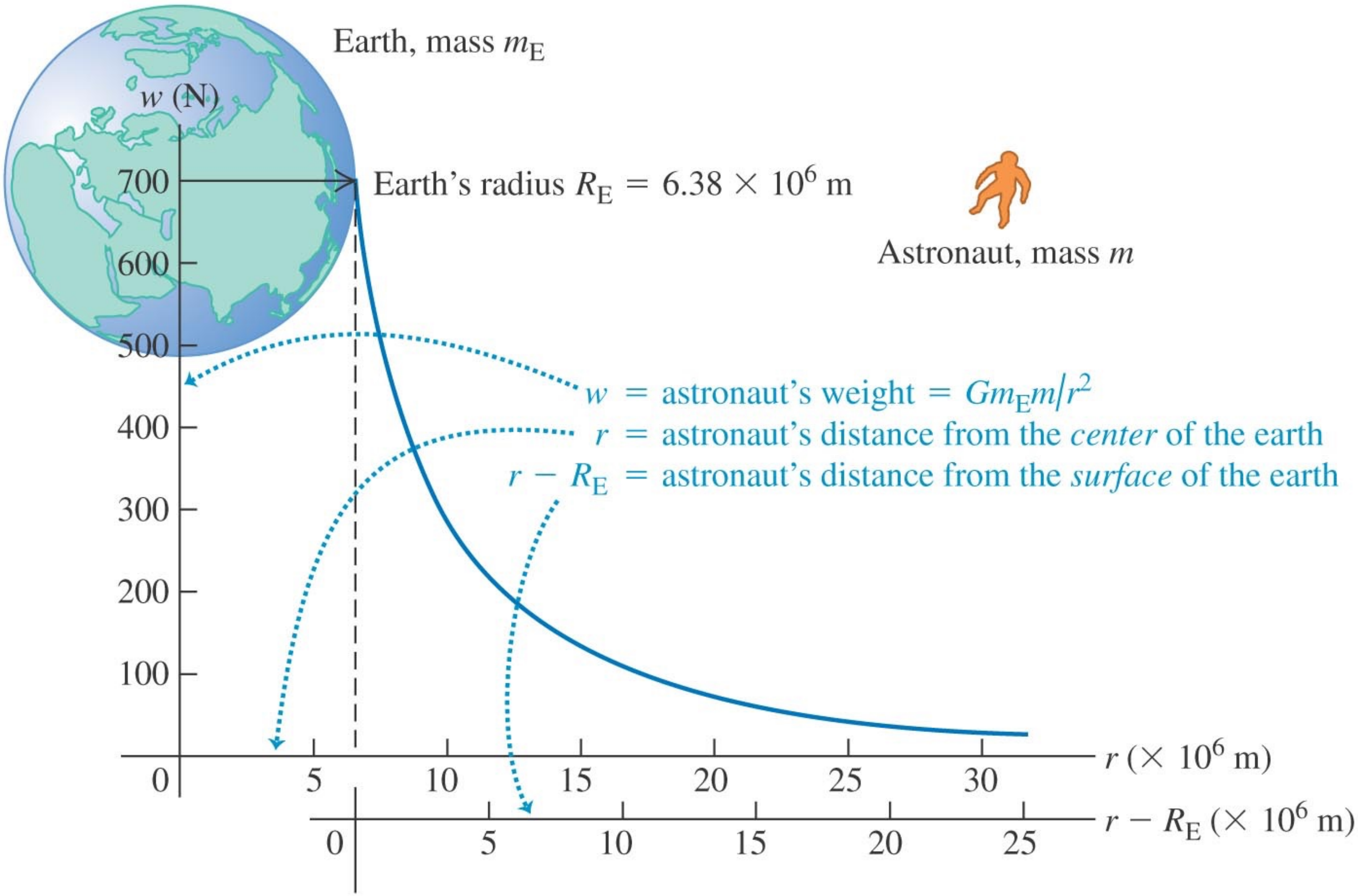
$w = mg$

- The acceleration due to gravity at the earth's surface is

$$g = Gm_E/R_E^2. \quad \approx 9.8 \ m/s^2$$

Weight

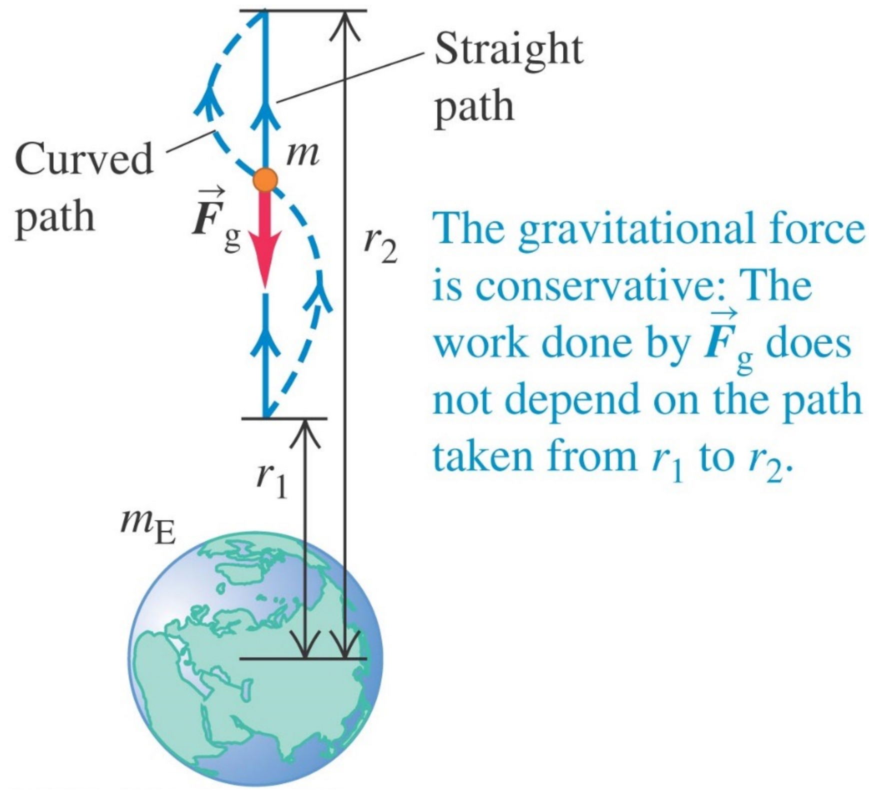
- The *weight* of a body decreases with its distance from the earth's center, as shown in Figure below.



Gravitational potential energy

mgh

- Follow the derivation of gravitational potential energy using Figure at the right.



- The *gravitational potential energy* of a system consisting of a particle of mass *m* and the earth is

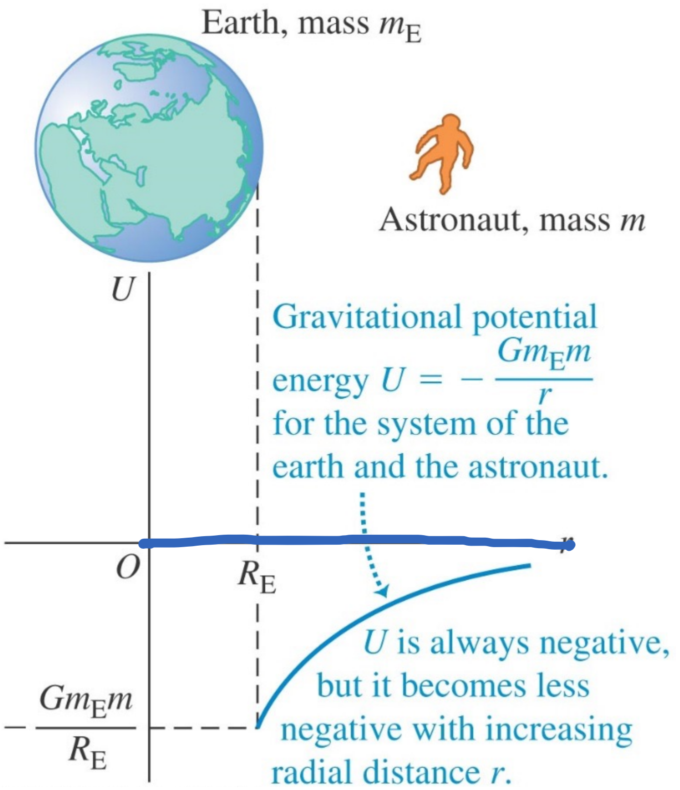
$$\boxed{U = -Gm_Em/r.}$$

$$U = -G\frac{m_E m}{r}$$

Δ

Gravitational potential energy depends on distance

- The gravitational potential energy of the earth-astronaut system *increases* (becomes less negative) as the astronaut moves away from the earth, as shown in Figure at the right.



From the earth to the moon

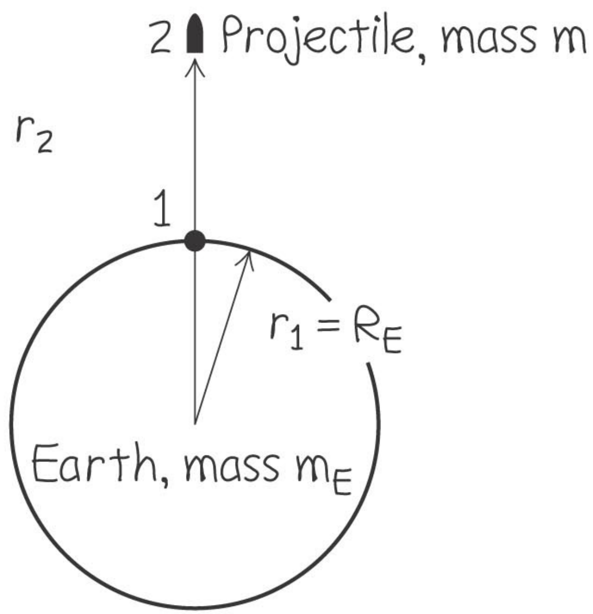
- To escape from the earth, an object must have the *escape speed.*

at ∞ : $U = 0$

$K = 0$

$U + K = 0$

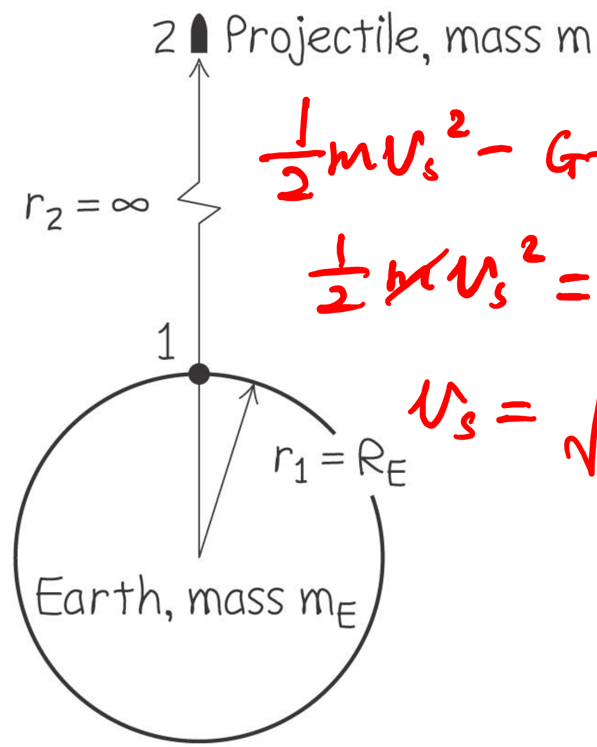
(a)



$K + U$

$= \frac{1}{2} m v_s^2 - G \frac{m_E m}{R_E}$

(b)



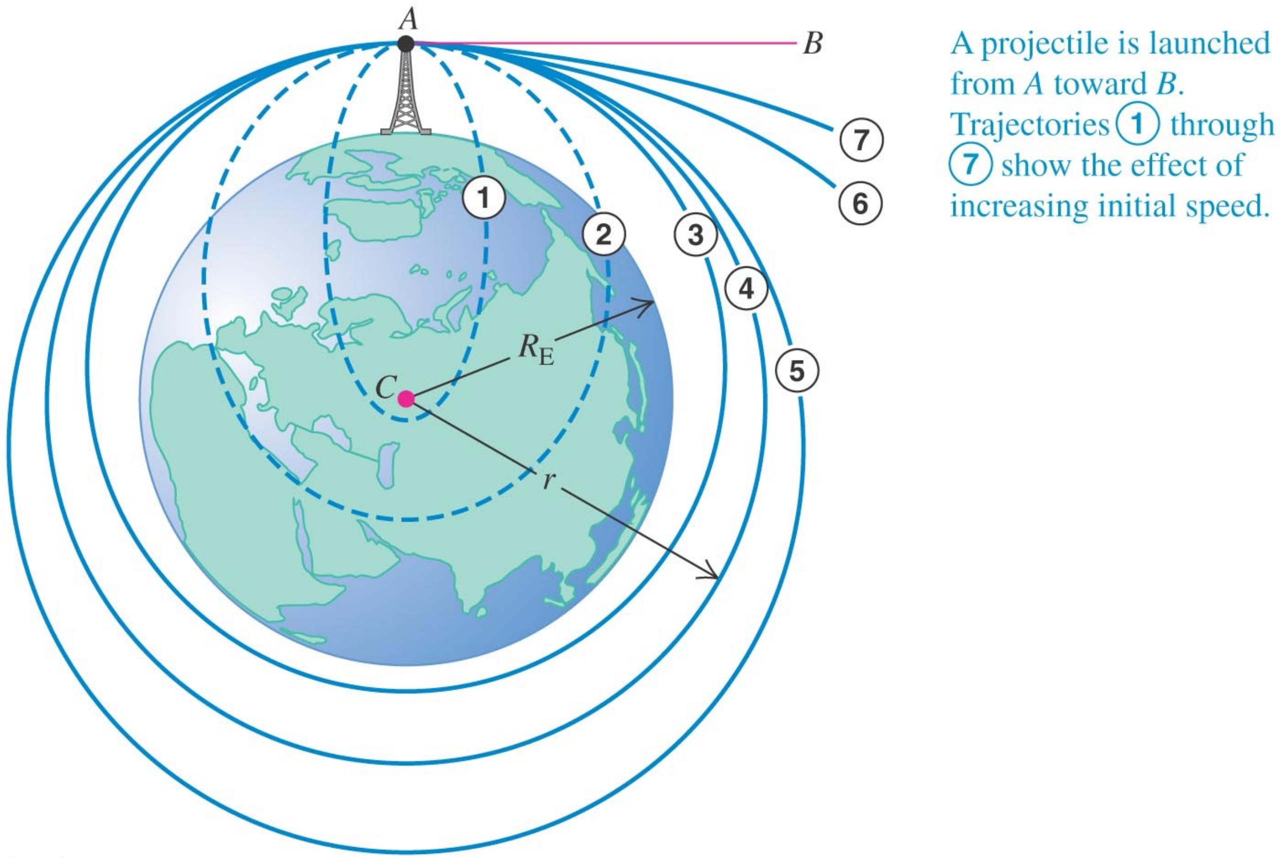
$\frac{1}{2} m v_s^2 - G \frac{m_E m}{R_E} = 0$

$\frac{1}{2} \cancel{m} v_s^2 = G \frac{m_E \cancel{m}}{R_E}$

$v_s = \sqrt{\frac{2 G m_E}{R_E}}$

The motion of satellites

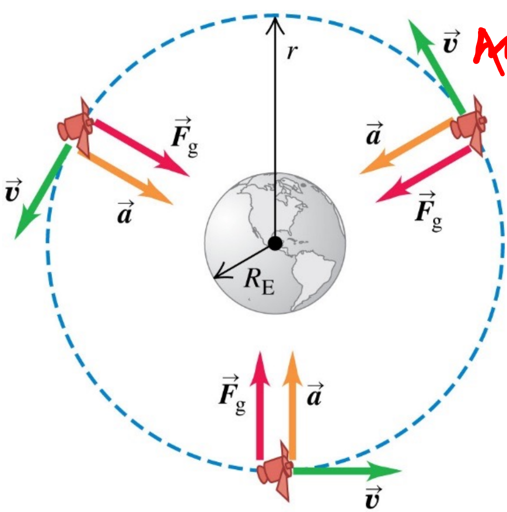
- The trajectory of a projectile fired from *A* toward *B* depends on its initial speed. If it is fired fast enough, it goes into a *closed elliptical orbit* (trajectories 3, 4, and 5 in Figure below).



Circular satellite orbits (why not falling)

- For a circular orbit, the speed of a satellite is just right to keep its distance from the center of the earth constant.
- A satellite is constantly falling *around* the earth. Astronauts inside the satellite in orbit are in a state of *apparent weightlessness* because they are falling with the satellite.

$a = v^2/r$, $F_g = ma = m\, v^2/r = F_g = G\frac{m_E m}{r^2}$



$\cancel{m}\frac{v^2}{\cancel{r}} = G\frac{m_E \cancel{m}}{r^{\cancel{2}}}$

$v = \sqrt{\frac{G m_E}{r}}$

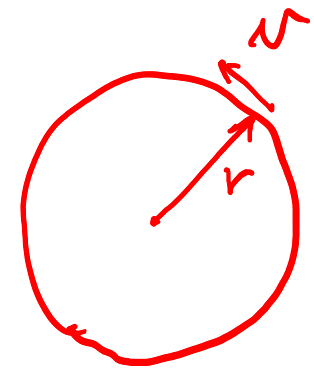
The satellite is in a circular orbit: Its acceleration $\vec{a}$ is always perpendicular to its velocity $\vec{v}$, so its speed v is constant.

Circular orbits

For a satellite in a circular orbit,

$a = v^2/r$

$$\frac{Gm_E m}{r^2} = \frac{mv^2}{r}$$



Then, the speed

$$v = \sqrt{\frac{Gm_E}{r}}$$

The period

$$T = \frac{2\pi r}{v} = \frac{2\pi r^{3/2}}{\sqrt{Gm_E}}$$

The total mechanical energy

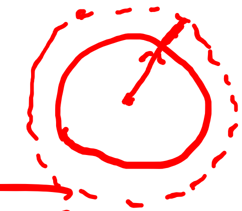
T = 24 hours

$$E = K + U = \frac{1}{2}mv^2 + \left(-\frac{Gm_E m}{r}\right) = -\frac{Gm_E m}{2r}$$

Example:

1. A satellite is launched into a circular orbit 750 km above the earth's surface. The mass of the satellite is 350 kg. The earth's radius is 6380 km. The mass of the earth is 5.97×10^{24} kg.
   - (a) What is magnitude of the gravitational force that the earth exerts on the satellite?
   - (b) What is the speed of the satellite in orbit?
   - (c) What is the gravitational potential energy of the earth-satellite system?

$$r = 6380\ km + 750\ km = 7130\ km$$



(a) $$F = G\frac{m_E m}{r^2} = 6.67\times10^{-11}\ N\cdot m^2/kg^2 \times \frac{5.97\times10^{24}\ kg \times 350\ kg}{(7130\ \cancel{km} \times 10^3\ m)^2}$$

$$= 2.7\times10^3\ N$$

(b) $$v = \sqrt{\frac{G m_E}{r}} = \sqrt{\frac{6.67\times10^{-11}\ N\cdot m^2/kg^2 \times 5.97\times10^{24}\ kg}{7130\times10^3\ m}}$$

$$= 7.47\times10^3\ m/s \sim 17000\ mi/h.$$

(c) $$U = -G\frac{m_E m}{r} = -6.67\times10^{-11}\ N\cdot m^2/kg^2 \times \frac{5.97\times10^{24}\ kg \times 350\ kg}{7130\times10^3\ m}$$

$$= - \quad J$$

Kepler's laws and planetary motion

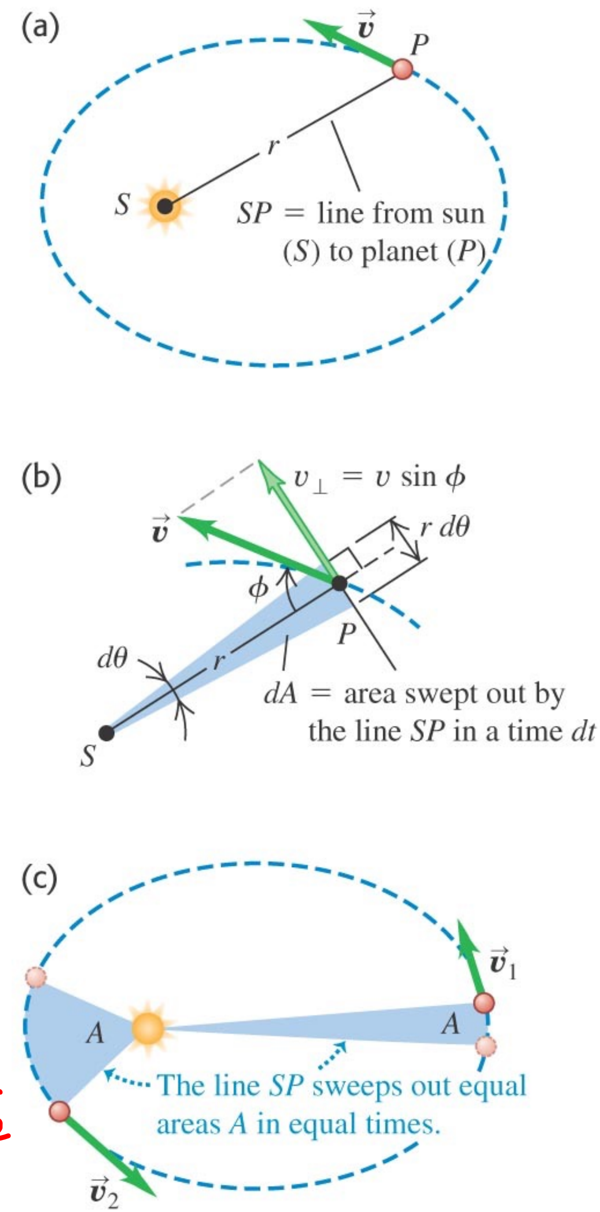
- Each planet moves in an elliptical orbit with the sun at one focus.
- A line from the sun to a given planet sweeps out equal areas in equal times (see Figure at the right).

$$\frac{dA}{dt} = \text{const.}$$

- The periods of the planets are proportional to the $^3/_2$ powers of the major axis lengths of their orbits.

$$T = \frac{2\pi a^{3/2}}{\sqrt{Gm_s}}$$

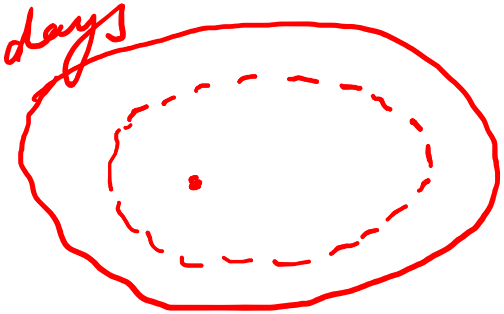
$T \propto a^{3/2}$
prop.



Example:

Planet Vulcan. Suppose that a planet (Vulcan) were discovered between the sun and Mercury and both Vulcan and Mercury are in elliptical orbits. If the major axis of Vulcan's orbit equal to 2/3 of the major axis of Mercury's orbit. The orbital period of Mercury is 88.0 days. What would be the orbital period of planet Vulcan?

$$a_v = \frac{2}{3} a_m \quad , \quad T_m = 88.0 \text{ days}$$



$$T \propto a^{3/2}$$

$$\frac{T_v}{T_m} = \frac{a_v^{3/2}}{a_m^{3/2}} = \left(\frac{a_v}{a_m}\right)^{3/2}$$

$$T_v = \left(\frac{a_v}{a_m}\right)^{3/2} \cdot T_m$$

$$= \left(\frac{2}{3}\right)^{3/2} \cdot 88.0 \text{ days}$$

$$= 0.54 \times 88.0 \text{ days}$$

$$= 47.9 \text{ days.}$$

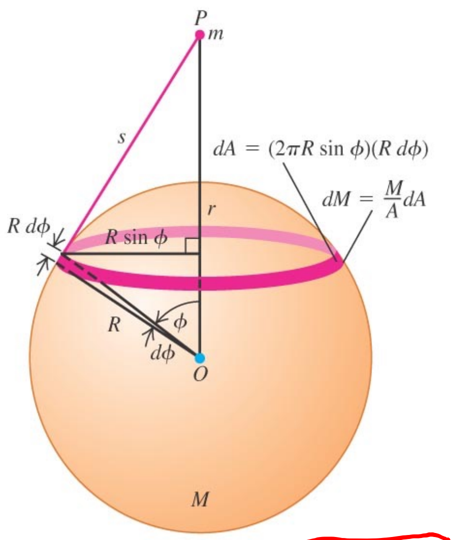
$$\left(\frac{2}{3}\right)^{3/2} = \left(\frac{2}{3}\right)^{3 \times \frac{1}{2}} = \left(\sqrt{\frac{2}{3}}\right)^{3}$$

Spherical mass distributions

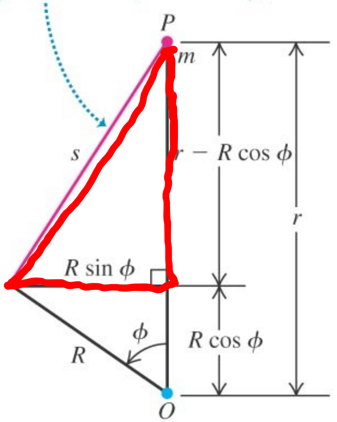
$F = -\nabla U$

- Prove that the gravitational interaction between two spherically symmetric mass distributions is the same as if each one were concentrated at its center.

(a) Geometry of the situation



(b) The distance s is the hypotenuse of a right triangle with sides $(r - R\cos\phi)$ and $R\sin\phi$.



shell: M, Area: $4\pi R^2$, surface density $\rho = \frac{M}{4\pi R^2}$

area of ring: $dA = 2\pi R\sin\phi \cdot R\,d\phi = 2\pi R^2 \sin\phi\,d\phi$

$$dM = \rho\,dA = \frac{M}{4\pi R^2} \cdot 2\pi R^2 \sin\phi\,d\phi = \frac{M}{2}\sin\phi\,d\phi$$

potential energy

$$du = -G\frac{m\,dM}{s} = -G\frac{Mm\sin\phi\,d\phi}{2s} \quad -①$$

$$s^2 = R^2\sin^2\phi + (r - R\cos\phi)^2$$

$$= R^2\sin^2\phi + r^2 - 2rR\cos\phi + R^2\cos^2\phi$$

$$s^2 = R^2 - 2rR\cos\phi + r^2$$

$$2s\,ds = 2rR\sin\phi\,d\phi$$

$$2s = \frac{2rR\sin\phi\,d\phi}{ds}$$

plug this into ①

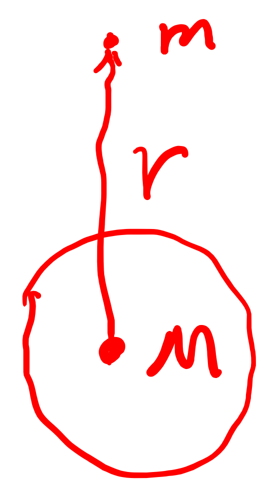
$$du = -\frac{GMm\,\cancel{\sin\phi}\,\cancel{d\phi}}{2rR\,\cancel{\sin\phi}\,\cancel{d\phi}}\,ds = -\frac{GMm}{2rR}\,ds$$

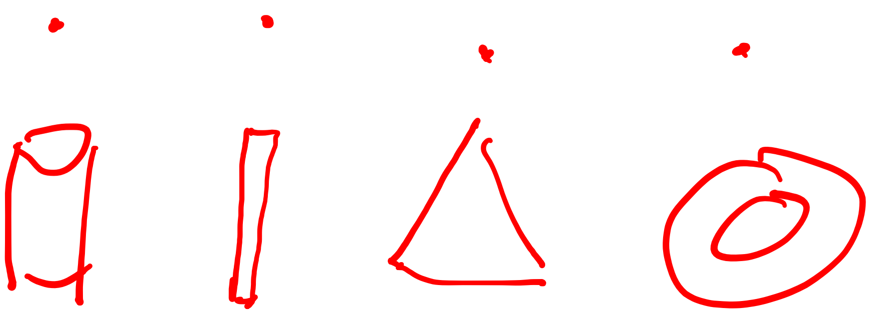
$$U = \int du = -\int_{r-R}^{r+R} \frac{GMm}{2rR}\,ds$$

$$= -\frac{GMm}{2rR}\int_{r-R}^{r+R} ds = -\frac{GMm}{2rR} \times 2R$$

$$U = -\frac{GMm}{r}$$

$$F = -\nabla U = -\frac{dU}{dr} = G\frac{Mm}{r^2}$$

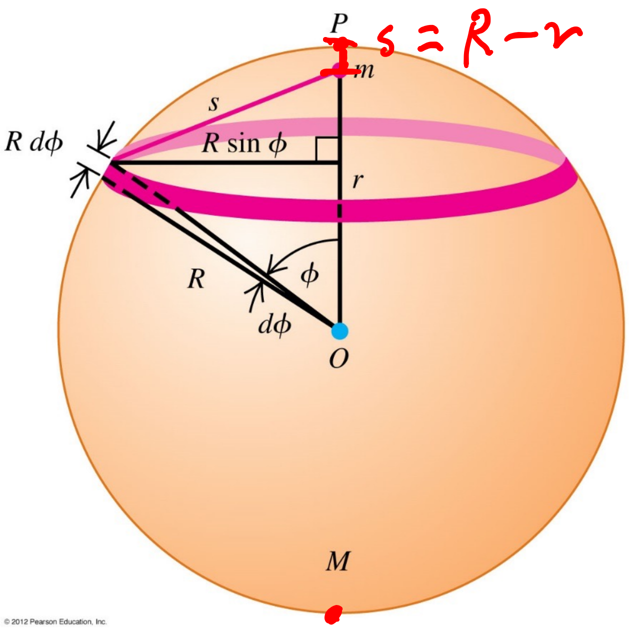




A point mass inside a spherical shell

- Inside the shell, the gravitational force exerted by the shell is zero.

$s = R - r$



$s = r + R$

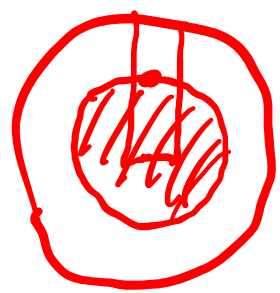
$$U = \int du = \int_{R-r}^{R+r} -G\frac{Mm}{2rR}\,ds$$

$$= -G\frac{Mm}{2rR}\int_{R-r}^{R+r} ds$$

$$= -G\frac{Mm}{\cancel{2r}R} \times \cancel{2r}$$

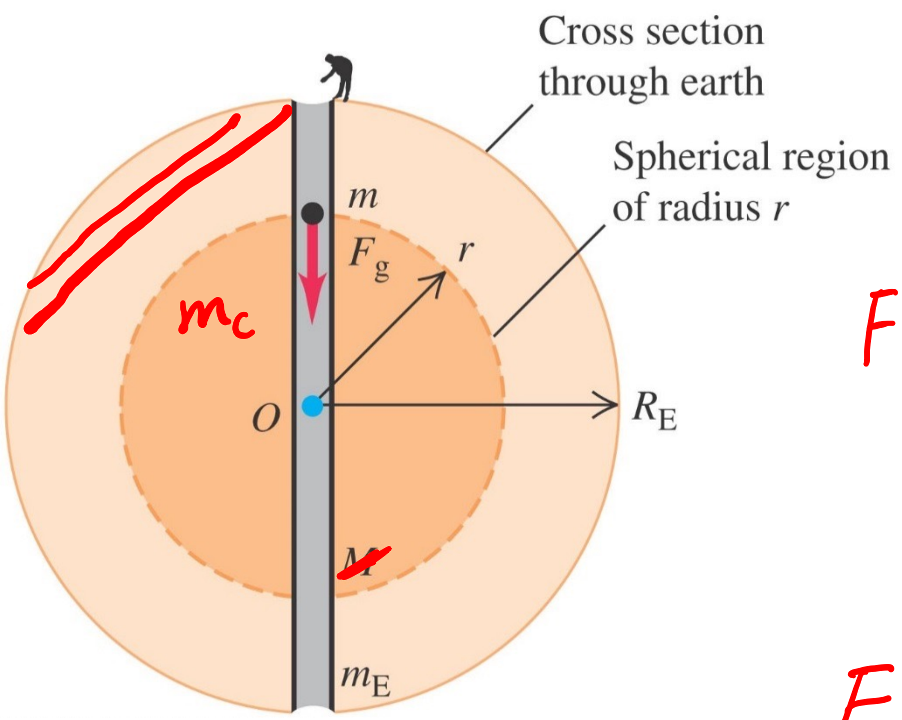
$$= -G\frac{Mm}{R}$$

$$F = -\nabla U = 0$$



A point mass inside a spherical shell

- If a point mass is inside a spherically symmetric shell, the potential energy of the system is constant. This means that the shell exerts no force on a point mass inside of it.



$$\rho = \frac{m_E}{V_E}$$

$$m_c = \rho V = \frac{m_E}{V_E} V = m_E \frac{\frac{4}{3}\pi r^3}{\frac{4}{3}\pi R_E^3}$$

$$m_c = m_E \frac{r^3}{R_E^3}$$

$$F = G\frac{m_c m}{r^2}$$

$$= G\frac{m\, m_E \frac{r^3}{R_E^3}}{r^2}$$

$$= G\frac{m\, m_E}{R_E^3} \cdot r$$

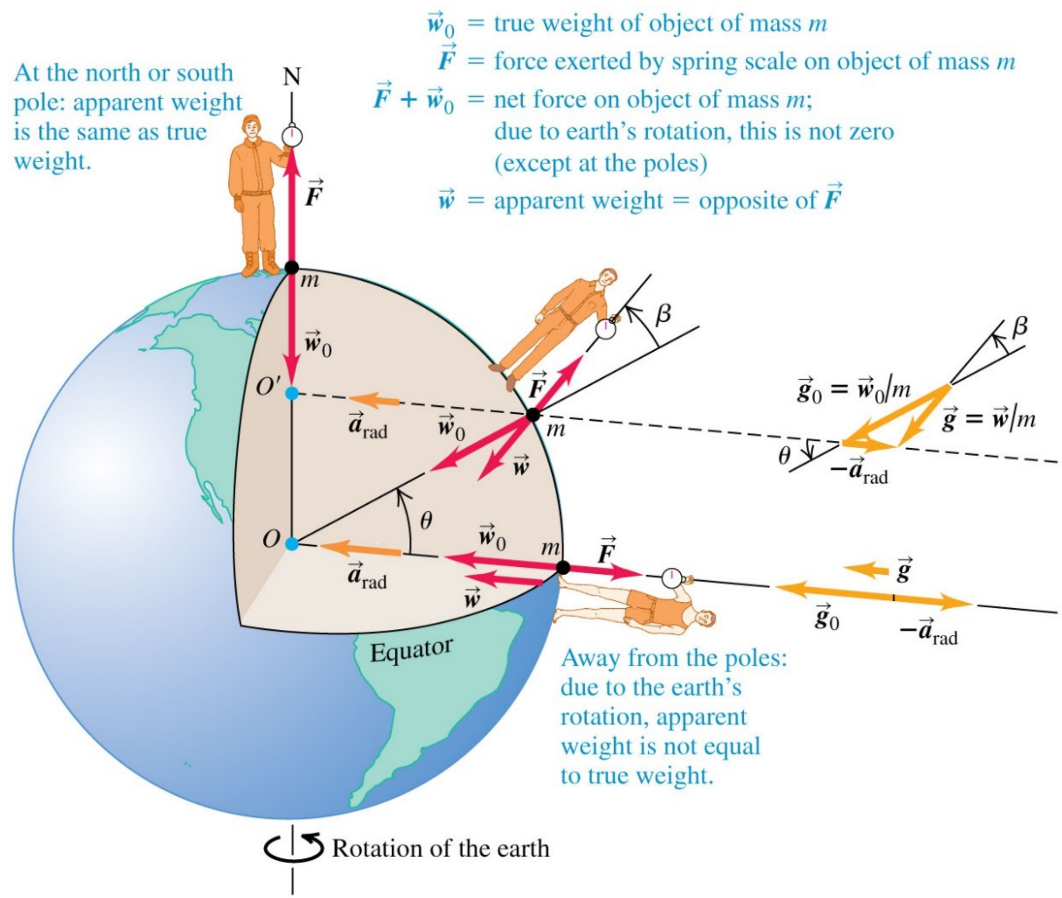
$$F = k \cdot r$$

The journey to the center of the earth

http://www.youtube.com/watch?v=LZYd19bIhP0

Apparent weight and the earth's rotation

- The *true weight* of an object is equal to the earth's gravitational attraction on it.
- The *apparent weight* of an object, as measured by the spring scale in Figure at the right, is less than the true weight due to the earth's rotation.



Black holes

$c = 3.0 \times 10^{8}$ m/s

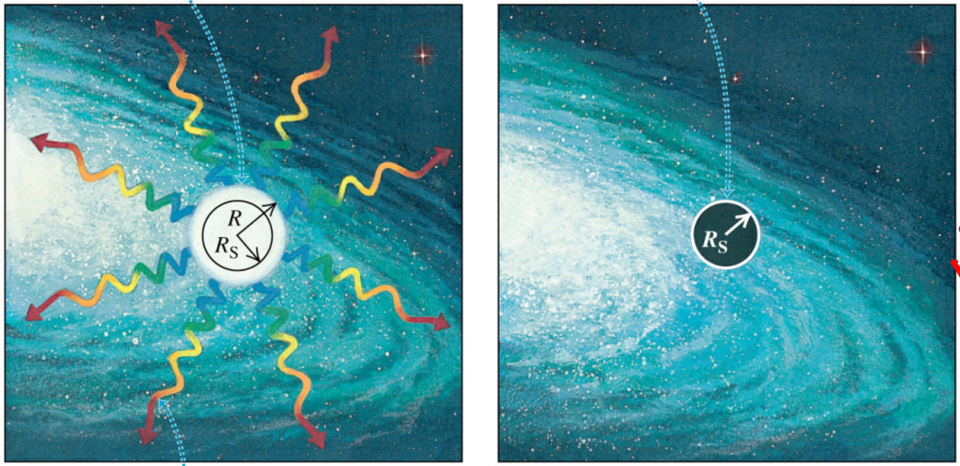
- If a spherical nonrotating body has radius less than the *Schwarzschild radius*, nothing can escape from it. Such a body is a *black hole*.
- The Schwarzschild radius is $R_S = 2GM/c^2$.
- The *event horizon* is the surface of the sphere of radius R_S surrounding a black hole.

$V_s = \sqrt{\frac{2GM}{r}}$

$\sqrt{\frac{2GM}{r}} = c$, $r = \frac{2GM}{c^2}$

(a) When the radius R of a body is greater than the Schwarzschild radius R_S, light can escape from the surface of the body.

(b) If all the mass of the body lies inside radius R_S, the body is a black hole: No light can escape from it.



Gravity acting on the escaping light "red shifts" it to longer wavelengths.

$r = \frac{2 \times 6.67 \times 10^{-11} \mathrm{N \cdot m^2/kg^2} \times 6 \times 10^{24} \mathrm{kg}}{(3.0 \times 10^{8} \mathrm{m/s})^2}$

$\approx \frac{8 \times 10^{14}}{9 \times 10^{16}}$ m

$\approx 10^{-2}$ m

$= 1$ cm