$\frac{\hbar^2}{2m}\frac{L^3}{r^2}$ Hydrogen $-\frac{\alpha}{r^0}$ $E = -\frac{Z^2\alpha^2 mc^2}{2n^2} = \frac{-\alpha^2 mc^2}{2n^2}$

PC - Quyaul 2012

7.3 Fine Structure Of The Hydrogen Atom

$\alpha = \frac{e^2}{\hbar c}$

7.3.1 The Spin-Orbit Coupling

$\alpha = \frac{e^2}{\hbar c}$

$\hbar L = r \times p$

$$J = L + S$$

$a = \frac{\hbar}{mc\alpha}$

$$\mu = \mu_{orb} + \mu_{spin} = -\mu_B L - g\mu_B S$$

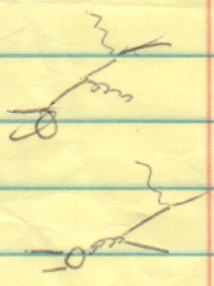
$$\boxed{\mu_B = \frac{\hbar e}{2mc}}$$

$$V = -\vec{\mu}\cdot\vec{B}$$

Magnetic Field due to protons Motion

proton charge q moving with velocity u

Biot-Savart



$$B = q\frac{u \times r}{c\, r^3}$$

Electric Field Due to Proton

$$E = \frac{q\, r}{r^3}$$

$$\vec{B} = -\frac{u \times \vec{E}}{c}.$$

$E = -\nabla\phi(r)$ (+)

$u = -V_{elect}$

$$\vec{B} = +\frac{V}{c}\times E = -\frac{V\times}{c}\nabla\phi(r)$$

(++)

1

$$B = -\nabla\phi(r) = -\left[i\frac{\partial}{\partial x}\phi(r) + j\frac{\partial}{\partial y}\phi(r) + \hat{k}\frac{\partial}{\partial z}\phi(r)\right]$$

$$= -\left[i\frac{\partial r}{\partial x}\frac{\partial\phi}{\partial r} + j\frac{\partial r}{\partial y}\frac{\partial\phi}{\partial r} + k\frac{\partial r}{\partial r}\frac{\partial\phi}{\partial r}\right]$$

$$= -\left\{i\frac{x}{r} + j\frac{y}{r} + k\frac{z}{r}\right\}\frac{\partial\phi}{\partial r} = -\frac{\vec{r}}{r}\frac{\partial\phi}{\partial r}$$

into — Magnetic Field seen by the electron at rest

$$B = -\frac{\vec{V}\times\vec{r}}{c\,r}\frac{\partial\phi}{\partial r} = +\frac{r\times P}{c\,m\,r}\frac{\partial\phi}{\partial r} = +\frac{L}{c\,m\,r}\frac{\partial\phi}{\partial r}$$

$$V_B = -\mu\cdot B = \frac{\mu_B \hbar L^2}{c\,m\,r}\frac{\partial\phi}{\partial r} + \frac{\mu_B g_e \hbar L\cdot S}{c\,m\,r}\frac{\partial\phi}{\partial r}$$

$$H_{LS} = +\frac{\mu_B g_e}{c\,m\,r}L\cdot S\frac{\partial\phi}{\partial r} = +\frac{\hbar^2 e}{2mc\;c\,m\,r}(L\cdot S)\frac{\partial\phi}{\partial r}$$

$V = -\frac{e}{r}$

$$\mu_B = \frac{\hbar e}{2mc}$$

$$H_{LS} = \frac{\hbar^2}{c^2}\frac{g}{2m^2 r}(L\cdot S)\frac{\partial e\phi}{\partial r} = \frac{\hbar^2}{c^2}\frac{g}{2m^2 r}(LS)\frac{dV}{dr}$$

$\Rightarrow$ $\boxed{\text{Experimental} \quad H_{LS} = \frac{\hbar^2}{c^2}\frac{g}{4m^2 r}(LS)\frac{dV}{dr}}$

2

But with $g = 2$ one obtains

$$H_{LS} = \frac{\hbar^2}{c^2}\frac{1}{2m^2 r}(LS)\frac{dV}{dr}$$

$$V = \frac{-eq}{r} \qquad \frac{dV}{dr} = + \frac{eq}{r^2}$$

$$\frac{kg\frac{m}{s} m}{kg^2 m^3}$$

$$H_{LS} = \frac{\hbar^2}{c^2} \frac{e^2}{2m^2 r^3}(LS) = \frac{\hbar^2}{c^2} \frac{e^2}{2m^2 r^3}(LS)$$

$$\frac{e^2}{\hbar c} = \alpha \qquad kg\frac{m}{s}$$

$$H_{LS} = \frac{\hbar}{c} \frac{\alpha}{2m^2 r^3} L \cdot S$$

$$\frac{kg\frac{m}{s} m}{kg^2 m^3}$$

$$kg \frac{MeV}{MeV^2 \frac{1}{MeV^3}} \qquad \frac{kg^2 \frac{m^4}{s^2}}{m/s} \qquad kg^2 \frac{m^3}{s} \frac{1}{kg\, m/s} \qquad \cancel{kg\, m} \qquad \frac{1}{kg\, m^3} \qquad \frac{kg\frac{m^2}{s}}{m/s} \qquad \frac{1}{kg\, m^3} \qquad kg\frac{m^2}{s^2}$$

7.3.2 Correction to Energy Levels

Eigenstates $|\phi^{ne}_{jm}\rangle$

$$\textcircled{*} \quad \Delta^{(1)} = \langle \phi^{ne}_{jm} | H_{LS} | \phi^{ne}_{jm} \rangle = \frac{\alpha}{2m^2} \langle \phi^{ne}_{jm} | \frac{1}{r^3} LS | \phi^{ne}_{jm} \rangle$$

I

$[LS, J] = 0$ Therefore

$$\textcircled{*} = \langle \phi^{ne}_{jm} | H_{LS} | \phi^{ne}_{jm} \rangle = \frac{\alpha}{2m^2} \langle \phi_{jm} | \frac{1}{r^3} | \phi_{jm} \rangle \times$$

$$I = \sum_{j'm'} |\phi_{j'm'}\rangle\langle\phi_{j'm'}| \qquad \langle \phi^{ne}_{jm} | LS | \phi^{ne}_{jm} \rangle$$

3

$$\left\langle \phi_{jm}^{nl} \left| \frac{1}{r^3} \right| \phi_{jm}^{nl} \right\rangle = \frac{\alpha^3 m^3}{n^3 \, l(l+1)(l+\frac{1}{2})}$$ $\not\propto j, m$

Now $\left\langle \phi_{jm}^{nl} \middle| LS \middle| \phi_{jm}^{nl} \right\rangle = \quad = //$

$$J = L + S \Rightarrow J^2 = L^2 + 2LS + S^2 \Rightarrow LS = \frac{J^2 - L^2 - S^2}{2}$$

$$// = \left\langle \phi_{jm}^{nl} \left| \frac{J^2 - L^2 - S^2}{2} \right| \phi_{jm}^{nl} \right\rangle =$$

$$\left\langle \phi_{jm}^{nl} \left| \frac{j(j+1) - l(l+1) - \frac{3}{4}}{2} \right| \phi_{jm}^{nl} \right\rangle = //$$

Possible values of j

(a) $\boxed{j = l + \frac{1}{2}}$

$$//(j = l + \tfrac{1}{2}) = (l + \tfrac{1}{2})(l + \tfrac{3}{2}) - l(l+1) - \tfrac{3}{4} =$$

$$= l^2 + \tfrac{3}{2}l + \tfrac{1}{2}l + \tfrac{3}{4} - l^2 - l - \tfrac{3}{4} = \boxed{l}$$

$2l \quad -l$

$j = l - \frac{1}{2}$

$$//(j = l - \tfrac{1}{2}) = (l - \tfrac{1}{2})(l + \tfrac{1}{2}) - l^2 - l - \tfrac{3}{4} = l^2 - \tfrac{1}{4} - l^2 - l - \tfrac{3}{4} = -(l+1)$$

$$\Delta_{LS} = \frac{\alpha}{2m^2} \frac{\alpha^3 m^3}{2n^3 \, l(l+1)(l+\frac{1}{2})} \begin{pmatrix} l \\ -(1+l) \end{pmatrix} \begin{matrix} j = l+\frac{1}{2} \\ j = l-\frac{1}{2} \end{matrix}$$

$P \quad L = \hbar n$

Not Complete - why?

Note on $l=0$ Not defined for Classical Picture

$\alpha^4 m$ -order

From Dirac equation $\rightarrow$

$\hookrightarrow$ Darwin term

$$H_3 = -\frac{1}{8m^2} \nabla^2 V(r)$$

$$V(r) = -\frac{\alpha}{r} \qquad \nabla^2\left(\frac{1}{r}\right) = -4\pi \delta_3(r)$$

$\rightarrow$ Contact Therm, Affects only S- states why?

$$E_3 = \frac{\alpha^4 m}{2n^3} \delta_{l0}$$

$\boxed{J = l + \frac{1}{2} \text{ - contains } E_3 \text{ term at } l=0}$

$$\Delta_{LS}^{j=l+\frac{1}{2}} = \frac{\hbar^2}{c^2} \frac{\alpha^4 m}{4n^3 (l+1)(l+\frac{1}{2})} \qquad \Bigg| \qquad E_n = -\frac{\alpha^2 m c^2}{2n^2}$$

7.3.3 The Relativistic Kinetic Energy Correction

Above correction was $\frac{v^2}{c^2}$ - effect.

Known as Relativistic Correction

$$T = E_{rel} - mc^2 = \sqrt{m^2c^4 + p^2c^2} - mc^2$$

$$= mc^2\sqrt{1 + \frac{p^2}{m^2c^2}} - mc^2 = \quad //$$

$$\sqrt{1+x}\Big|_0 = 1 + \frac{1}{2\sqrt{1+x}}\Big|_{x=0} x - \frac{1}{2} \cdot \frac{1}{2}\frac{1}{2}\frac{1}{(1+x)^{3/2}} x^2$$

$$= 1 + \frac{1}{2}x - \frac{1}{8}x^2$$

$$// = mc^2\left(1 + \frac{1}{2}\frac{p^2}{m^2c^2} - \frac{1}{8}\frac{p^4}{m^4c^4}\right) - mc^2 =$$

$$= \frac{1}{2}\frac{p^2}{m} - \frac{1}{8}\frac{p^4}{m^3c^2} = \frac{p^2}{2m} - \frac{1}{8}\frac{p^4}{m^3c^2}$$

$\frac{p^2}{c^2} \sim \frac{v^2}{c^2}$

~~Fact~~ relativistic effect

$$H_{kin} = -\frac{1}{2m}\frac{p^4}{4m^2c^2} = -\frac{1}{2m}T_0^2 = -\frac{1}{2m}(H_0 - V)^2$$

$$\langle \phi_{jm}^{n,l} | H_{kin} | \phi_{jm}^{n,l} \rangle =$$

$$= -\frac{1}{2m} \langle \phi_{jm}^{n,l} | H_0^2 - VH_0 - H_0V + V^2 | \phi_{jm}^{nl} \rangle$$

Conside $\boxed{\langle \phi_{jm}^{nl} | H_0^2 | \phi_{jm}^{nl} = E_n^{0\,2}}$

$$\langle \phi_{jm}^{nl} | H_0 V | \phi_{jm}^{nl} = \langle \phi_{jm}^{nl} | H_0 | \phi_{jm}^{nl} \rangle \langle \phi_{jm}^{nl} | V | \phi_{jm}^{nl} \rangle$$

$$= + E_n^0 \langle \phi_{jm}^{nl} | V | \phi_{jm}^{nl} \rangle$$

$$\langle \phi_{jm}^{nl} | V | \phi_{jm}^{nl} \rangle = ?$$

Virial Theorem

$$2\langle T \rangle = \langle \vec{r}\, \nabla V \rangle$$

~~$\nabla V = -\alpha \nabla^2 \frac{1}{r}$~~

7 $$2\langle \phi_{jm}^{nl} | T | \phi_{jm}^{nl} \rangle = \langle \phi_{jm}^{nl} | \vec{r}\,\vec{\nabla} V | \phi_{jm}^{nl} \rangle$$

$$\vec{\nabla} V = -\alpha \vec{\nabla}\frac{1}{r} = \alpha\frac{\vec{r}}{r}\partial\frac{1}{r} = \alpha\frac{\vec{r}}{r^3}$$

$$2\langle\phi|T|\phi\rangle = \alpha\langle\phi|\frac{r^2}{r^3}|\phi\rangle =$$

$$= \langle\phi|\frac{\alpha}{r}|\phi\rangle =$$

$$= -\langle\phi|V|\phi\rangle$$

$$2\langle\phi|T+V|\phi\rangle - 2\langle\phi|V\rangle = -\langle|V|\rangle$$

$$2\langle\phi|H_0|\phi\rangle = \langle|V|\rangle$$

$$2E_n^0 = \langle|V|\rangle \quad | \quad \langle|V\rangle = 2E_n^0$$

$$\boxed{\langle\phi_{jm}^{nl}|H_0V|\phi_{jm}^{nl}\rangle = 2E_n^0}$$

Need HW

$$\boxed{\langle\phi_{jm}^{nl}|V^2|\phi_{jm}^{nl}\rangle = \frac{\alpha^2}{a^3n^3(l+\frac{1}{2})}}$$

$a = \frac{\hbar}{mc\alpha}$

$$\boxed{\frac{\alpha^4m^2}{\cancel{\ }n^3(l+\frac{1}{2})}}$$

8

$$\Delta_{kin} = -\frac{1}{2m}\left(E_n^{0\,2} - 4E_n^{0\,2} + \frac{\alpha^4 m^2}{\cancel{m} n^3(l+\frac{1}{2})}\right)$$

$$= -\frac{1}{2m}\left(-3E_n^{(0)\,2} + \frac{\alpha^4 m^2}{\cancel{m} n^3(l+\frac{1}{2})}\right)$$

$$E_n^{(0)} = -\frac{\alpha^2 m}{2n^2}$$

$$\boxed{\Delta_{kin} =} -\frac{1}{2m}\left(\frac{-3\alpha^4 m^2}{4n^4} + \frac{\alpha^4 m^2}{\cancel{m} n^3(l+\frac{1}{2})}\right) =$$

$$= \boxed{\frac{\alpha^4 m}{2n^4}\left(\frac{3}{4} - \frac{n}{(l+\frac{1}{2})}\right)}$$

Fine Structure Constant

$$\Delta_{FS} = \Delta_{KM} + \Delta_{LS} =$$

$$= \frac{\alpha^4 m}{2n^3\, l(l+\frac{1}{2})(l+\frac{1}{2})} \begin{pmatrix} l \\ -(1+l) \end{pmatrix} + \frac{\alpha^4 m}{2n^4}\left(\frac{3}{4} - \frac{n}{l+\frac{1}{2}}\right)$$

$$= \frac{\alpha^4 m}{2n^4}\left[\frac{n}{2\, l(l+\frac{1}{2})(l+\frac{1}{2})} \begin{pmatrix} l \\ + \\ -(1+l) \end{pmatrix} \frac{3}{4} - \frac{n}{l+\frac{1}{2}}\right] = 11$$

(a) $j = l + \frac{1}{2}$ $\qquad l = j - \frac{1}{2}$

$$11^a = \frac{\alpha^4 m}{2n^4}\left[\frac{n}{2(j-\frac{1}{2})\, j\, (j+\frac{1}{2})}\, j-\tfrac{1}{2} + \frac{3}{4} - \frac{n}{j}\right]$$

$$= \frac{\alpha^4 m}{2n^4}\left[\frac{n}{2(j+\frac{1}{2})j} + \frac{3}{4} - \frac{n}{j}\right] =$$

10 $$= \frac{\alpha^4 m}{2n^4}\left[\frac{n(1 - 2j + 1)}{2j(j+\frac{1}{2})} + \frac{3}{4}\right] = \frac{\alpha^4 m}{2n^4}\left[\frac{3}{4} - \frac{2n}{2j+1}\right]$$

(b) $j = l - \frac{1}{2} \quad l = j + \frac{1}{2}$

$$11^{B} = \frac{\alpha^4 m}{2n^4}\left[\frac{n \;{}^{-(j+\frac{3}{2})}}{2(j+\frac{1}{2})(j+\frac{3}{2})(j+1)} + \frac{3}{4} - \frac{n}{j+1}\right]$$

$$= \frac{\alpha^4 m}{2n^4}\left[-\frac{n}{2(j+\frac{1}{2})(j+1)} - \frac{n}{j+1} + \frac{3}{4}\right]$$

$$= \frac{\alpha^4 m}{2n^4}\left[\frac{-n(2j+1+1)}{(2j+1)(j+1)} + \frac{3}{4}\right] = \frac{\alpha^4 m}{2n^4}\left[\frac{3}{4} - \frac{2n}{2j+1}\right]$$

$$\Delta_{FS} = \frac{\alpha^4 m}{2n^4}\left(\frac{3}{4} - \frac{2n}{2j+1}\right)$$

Splitting depends on j $\quad l \leq n-1$

$2P_{\frac{1}{2}}$ $\quad 2P_{\frac{3}{2}}$ — split

11 $2S_{\frac{1}{2}}$ $\quad 2P_{\frac{1}{2}}$ — not Lamb shift

Splitting between $J=\frac{1}{2}$ $J=\frac{3}{2}$

$$\Delta = \frac{\alpha^4 m}{4n^3}$$

For $n=2$ $\Delta = 4.528 \times 10^{-5}\,eV$

$J=\frac{1}{2}$ $\frac{3}{4} - \frac{4}{2} = \frac{3}{4} - 2 = -\frac{5}{4}$

$J=\frac{3}{2}$ $\frac{3}{4} - \frac{4}{4} = \frac{3}{4} - 1 = -\frac{1}{4}$

Gumprecht 2012

7.3.4. The Fine Structure of the Hydrogen Atom

$$\Delta_{FS} = \frac{\alpha^4 m}{2n^4}\left(\frac{3}{4} - \frac{2n}{2j+1}\right)$$

$j = \ell + \frac{1}{2}$

$j = \ell - \frac{1}{2}$

$\ell < n-1$

Splitting Between $j=\frac{1}{2}$ $j=\frac{3}{2}$

$$\Delta_{FS}^{j=\frac{1}{2}} = \frac{\alpha^4 m}{2n^4}\left(\frac{3}{4} - \frac{2n}{2}\right) = \frac{\alpha^4 m}{2n^4}\left(\frac{3}{4} - n\right)$$

$$\Delta_{FS}^{j=\frac{3}{2}} = \frac{\alpha^4 m}{2n^4}\left(\frac{3}{4} - \frac{2n}{3+1}\right) = \frac{\alpha^4 m}{2n^4}\left(\frac{3}{4} - \frac{n}{2}\right)$$

$$\Delta_{FS}^{j=\frac{3}{2}} - \Delta_{FS}^{j=\frac{1}{2}} = \frac{\alpha^4 m}{2n^3}\frac{1}{2} = \frac{\alpha^4 m}{4n^3}$$

For $n=2$ $\Delta_{FS}^{\frac{3}{2}-\frac{1}{2}} = \frac{\alpha^4 m}{32} = \boxed{4.528\times10^{-5}\,eV}$

$10^6\,eV$

$0.52\,MeV$

5.2×10^5

520000

$n=1 \to n=2$ Lyman α

$10.2\,eV$

1

For $\boxed{n=1}$ state $j = \frac{1}{2}$

$$\Delta_{FS} = \frac{\alpha^4 m}{2}\left(\frac{3}{4} - 1\right) = -\frac{\alpha^4 m}{2}\frac{1}{4} = -\frac{\alpha^4 m}{8}$$

$$\boxed{\Delta_{FS}^{n=1} = 1.13\ 10^{-5}\ eV}$$

$$E_n = -\frac{\alpha^2 m}{2n^2} \qquad E_1 = E_g = -\frac{\alpha^2 m}{2} = 13.6\,eV$$

$m_e = 510998.9\,eV$

Darwin term

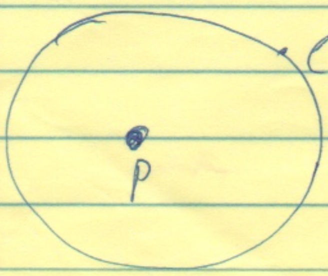
$E_3 = \frac{\alpha^4 m}{2n^3}$

$$H_3 = -\frac{1}{8m^2}\nabla^2(V) \qquad \nabla^2\left(\frac{1}{r}\right) = -4\pi\delta_3 r$$

only s state

7.3.6 The Hyperfine Structure of the Hydrogen Atom

$$H_0 = \frac{p^2}{2m} - \frac{\alpha}{r}$$



p – has a spin

$$\mu_p = \underline{g_p}\, \mu_{B,N}\, S_p \rightarrow \text{proton spin}$$

$$\mu_{B,N} = \frac{e}{2M} \rightarrow 938.272\ MeV$$

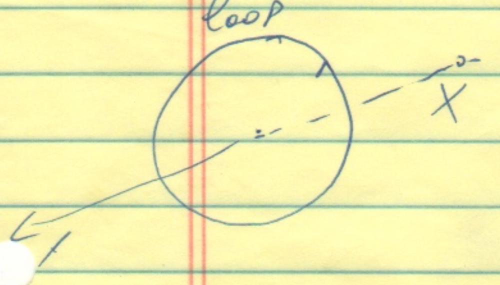
g_p: 2×2.793

Total Spin of the System

$$F = L + S_e + S_p$$

↳ Truly Conserved Quantity

⇒ Magnetic Field of the Dipole



$\mu_{Loop} = I\pi a^2$

$x \gg a$

$$B_x = \frac{\mu_0 I a^2}{2(x^2+a^2)^{3/2}} \simeq \frac{\mu_0 I a^2}{2x^3} = \frac{\mu_0 \mu_{Loop}}{2\pi X^3}$$

$$B_x = \frac{\mu_0 \mu_{Loop}}{2\pi X^3} = (\nabla \times A)_x = \nabla_y A_z - \nabla_z A_y$$

$\mu_{Loo \to X}$

$$\boxed{A = -\vec{\mu}_{Loop} \times \nabla \cdot \frac{1}{r}} \text{ — ucu}$$

$$A_z = -\mu_x \cdot \nabla_y\left(\cdot \frac{1}{r}\right) = \mu_x\left(\frac{y}{r^3}\right)$$

$\left(\sqrt{x^2 y^2 + z^2}\right)^{\frac{3}{2}}$

$$A_y = +\mu_x \nabla_z\left(\frac{1}{r}\right) = -\mu_x\left(\frac{z}{r^3}\right)$$

x y z

$$\nabla_y A_z = \mu_y\left(\frac{1}{r^3} - 3\frac{y^2}{r^5}\right)$$

3

$$\nabla_z A_y = -\mu_x\left(\frac{1}{r^3} - 3\frac{z^2}{r^5}\right)$$

$$\boxed{\vec{B} = \vec{\nabla}\times\vec{A} \qquad A = -\vec{\mu}_p \times \nabla\frac{1}{r}}$$ ucr

$B(A\cdot C) - C(A\cdot B)$

$$\boxed{B} = -\nabla\times\left(\vec{\mu}_p \times \left(\nabla\frac{1}{r}\right)\right) = -\vec{\mu}_p\, \nabla\cdot\left(\nabla\frac{1}{r}\right) + \nabla\cdot\vec{\mu}_p\left(\vec{\nabla}\frac{1}{r}\right)$$

$$= \boxed{\left(\vec{\mu}_p\cdot\vec{\nabla}\right)\left(\vec{\nabla}\frac{1}{r}\right) + \vec{\mu}_p \cdot \nabla^2\frac{1}{r}}$$

Next page

$$B_z = (\mu_{p_z} \nabla_z)\left(\nabla_z\frac{1}{r}\right)$$

How to include A_μ into the langragian $\boxed{\text{orbital}}$

$$H_0 = \frac{P^2}{2m} \to \frac{1}{2m}(P + eA)^2 = \frac{P^2}{2m} + H'_{orbital}$$

$$H'_{orbital} = \frac{1}{2m}\left(\vec{P} - e\mu_p \times \nabla\frac{1}{r}\right)^3 - \frac{P^2}{2m} =$$

$$= \frac{e}{m}\, P\cdot\left(\overrightarrow{\nabla\left(\frac{1}{r}\right)\times\mu_p}\right) = -\frac{e}{m}\,\vec{P}\cdot\left(\frac{\vec{r}}{r^3}\times\vec{\mu}\right) =$$

$A\cdot(B\times C) =$

$$= -\frac{e}{m}\frac{1}{r^3}\vec{\mu}_p\cdot\left(\vec{P}\times\vec{r}\right) = +\frac{e}{m}\frac{1}{r^3}\vec{\mu}_p\left(\vec{r}\times\vec{P}\right)$$

$$= +\frac{e}{m}\frac{1}{r^3}\vec{\mu}_p\cdot\vec{L} = +\frac{g_D\,\alpha}{2Mm}\frac{1}{r^3}S_p\cdot L$$

$$\mu_p = \frac{|e|\,g_D}{2M}$$

$\mu_e = -\frac{e}{}$

$\mu_p \quad S_e$ interaction

$$H'_{spin} = -\vec{\mu}_e \cdot \vec{B} = \frac{ge}{2m} \vec{S} \cdot \vec{B} \Rightarrow U =$$

$$B = \vec{\mu}_p \cdot \vec{\nabla} \left(\vec{\nabla} \frac{1}{r} \right) - \mu_p \nabla^2 \frac{1}{r}$$

$$= U = \frac{g_e e}{2m} \frac{g_p e}{2M} \left[(S_p \nabla)(S_e \nabla) \frac{1}{r} - S_p S_e \nabla^2 \frac{1}{r} \right]$$

$$= \frac{g g_p \alpha}{4mM} \sum_{ij} S_{ei} S_{pj} T_{ij}(r)$$

$$\boxed{\begin{aligned} T_{ij} &= (\partial_i \partial_j - \delta_{ij} \nabla^2)\left(\frac{1}{r}\right) = \\ &= \frac{1}{3} \delta_{ij} \overset{+}{\sum_k} T_{kk} + \left(T_{ij} - \frac{1}{3} \delta_{ij} \sum_k T_{kk} \right) \end{aligned}}$$

$$\sum_k T_{kk} \left(\nabla^2 - 3\nabla^2 \right) \frac{1}{2} = -2 \nabla^2 \left(\frac{1}{r} \right)$$

$$T_{ij} = -\frac{2}{3} \delta_{ij} \nabla^2 \left(\frac{1}{r} \right) + \underbrace{\partial_i \partial_j - \delta_{ij} \nabla^2 + \frac{2}{3} \nabla^2}_{\left(\partial_i \partial_j - \frac{1}{3} \delta_{ij} \nabla^2 \right) \frac{1}{r}}$$

5

$$T_{ij} = -\frac{2}{3}\delta_{ij}\nabla^2\left(\frac{1}{r}\right) + \left(\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2\right)\frac{1}{r}$$

using $\boxed{\nabla^2\left(\frac{1}{r}\right) = -4\pi\delta(r)}$

does not have diagonal term

$$H'_{spin} = H'_{contact} + H'_{dipole}$$

$$H'_{contact} = \frac{g g_p \alpha}{4m\pi}\frac{8\pi}{3} s_e \cdot s_p \, \delta^3(r)$$

$$H'_{dipole} = \frac{g g_p \alpha}{4mM} \sum_{i,j} s_{ei} s_{pj} \left(\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2\right)\frac{1}{r}$$

$$= \frac{g g_p \alpha}{4mM}\frac{1}{r^5} \sum_{ij} s_{ei} s_{pj} \left(3 r_i r_j - \delta_{ij} r^2\right)$$

Hyperfine Splitting of Ground State

$n=1 \quad l=0 \qquad F = S_p + S_n$

$$F = \begin{matrix} 1 \\ 0 \end{matrix}$$

For $s=0$ $l=0$ Orbital Term Does not contribut HW

Also $\langle H'_{dipole} \rangle = 0$ HW

Only Contact Term is Left

$$\Delta^{cont} = \frac{g g_p \alpha}{4mM} \frac{8\pi}{3} S_e \cdot S_p \int \delta^3(r) \left| \phi_{100}(r) \right|^2 d^3 r$$

$$R_{10} = \frac{2}{a^{3/2}} e^{-ra} \qquad \phi_{100} = R_{10} Y_{00} = \frac{2}{a^{3/2}} e^{-ra} \frac{1}{\sqrt{4\pi}}$$

$a = \frac{1}{m\alpha}$ $\qquad \phi_{100} =$

$$\Delta^{cont} = \frac{g g_p \alpha}{4mM} \frac{8\pi}{3} \left| \phi_{100}(0) \right|^2 S_e S_p$$

$$\Delta^{cont} = \frac{g g_e \alpha}{4mM} \frac{8\pi}{3} \frac{1}{4\pi} \frac{4}{a^3} = \frac{g g_e \alpha}{4mM} \frac{2}{3} \cdot m^3 \alpha^3 \, 4 \, S_e S_p$$

$$\delta^{cont} = \frac{g_e \alpha^4}{3} \frac{m^2}{M} \left(2 S_e S_p\right)$$

$$(F)^2 = S_e^2 + 2 S_e S_p + S_p^2 = \frac{3}{2} + 2 S_e S_p$$

$$2 S_e S_p = f(f+1) - \frac{3}{2}$$

$f = 1$

$2 - \frac{3}{2}$

$f = 0$

$$2 S_e S_p = \begin{pmatrix} \frac{1}{2} \\ -\frac{3}{2} \end{pmatrix}$$

$$\Delta_{cont} = \Delta_{HFS} = \frac{\alpha^4}{3} \frac{m}{M} g g_p m \begin{pmatrix} \frac{1}{2} \\ -\frac{3}{2} \end{pmatrix} \quad F = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

splitting

$$\delta^{F=1\,(\text{spins - parallel})} - \delta^{F=0\,(\uparrow\downarrow,\ \text{most bound})} =$$

$$= \frac{2}{3} \cdot \alpha^4 \cdot \frac{m}{M} g g_p m = \frac{2}{3} \frac{1}{137^4} \times \frac{1}{1836} \times 2 \cdot 2 \times 2.78 \cdot 511.60$$

$$\Delta \boxed{= 5.85005\ 10^{-6}\ eV}$$

$$\gamma \quad \lambda = \frac{2\pi}{\Delta} = \frac{2\pi \cdot 1.97 \times 10^{-5}\ eV\text{-}cm}{5.86 \times 10^{-6}\ eV} \simeq 21.2\ cm$$

Discovered in 1951

used to prove the spiral structure of