HyperFine

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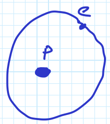
Hyperfine Structure of Hydrogen Atom



$$H_0 = \frac{p^2}{2m_e} - \frac{\alpha}{r}$$

- But proton has spin too
- Thus it has also magnetic moment

$$\mu_p = g_p \frac{e}{2M_p} \vec{S}_p$$

$$M_p = 938.272\ \mathrm{MeV}$$

- g_p - if proton is elementary particle it $= 2$
  - however $g_p = 2 \times 2.793$
    Anomalous Magnetic Moment

- Total Spin of Nucleus

$$\vec{F} = \vec{L} + \vec{S}_e + \vec{S}_p$$

  └ Truly Conserved quantity

- Magnetic Moment also Creates Magnetic Field

From Classical Electromagnetism

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

- Vector Potential of the current loop loop at infinitely far

$$\vec{A} = -\vec{\mu} \times \vec{\nabla} \frac{1}{r}$$

- We adopt it for Proton

$$\vec{A}_p = -\vec{\mu}_p \times \vec{\nabla} \frac{1}{r}$$

Therefor $\vec{B}_{sp} = -\vec{\nabla} \times \left(\vec{\mu}_p \times \vec{\nabla} \frac{1}{r}\right)$

- Under such magnetic field Electron will feel the interation

$$H = -\mu_e B_{sp}$$

$$H' = \frac{g e}{2m} \vec{S} \cdot \vec{B}_{sp} = \frac{g_e}{2m} \frac{g_p e}{2M_p} \Big[(-1) \vec{S}_e \underbrace{\left(\vec{\nabla} \times \vec{S}_p \times \vec{\nabla} \frac{1}{r} \right)}_{\text{Term}} \Big]$$

Consider $-\overset{A}{\vec{\nabla}} \times \overset{B}{\vec{S}_p} \times \overset{C}{\vec{\nabla}} \frac{1}{r} = -\left(\overset{B(A\ \ C)}{\vec{S}_p \left(\vec{\nabla} \cdot \left(\vec{\nabla} \frac{1}{r} \right) \right)} - \overset{(A B)\cdot}{\left(\vec{\nabla} \cdot \vec{S}_p \right)} \vec{\nabla} \frac{1}{r} \right)$

$$// = \left(\vec{S}_p \cdot \vec{\nabla} \right) \vec{\nabla} \frac{1}{r} - \vec{S}_p \cdot \nabla^2 \frac{1}{r} \Big) = \text{Term}$$

$$H' = \frac{g g_p e^2}{4 m_e M_p} \Big[\underbrace{\left(\vec{S}_p \cdot \vec{\nabla} \right)}_{\sum_i S_p^i \nabla_i} \underbrace{\left(\vec{S}_e \cdot \vec{\nabla} \right)}_{\sum_j S_e^j \nabla_j} \frac{1}{r} - \vec{S}_e \cdot \vec{S}_p \nabla^2 \frac{1}{r} \Big] =$$

$$= \frac{g g_p \alpha}{4 m_e M_p} \Big[\sum_{ij} S_p^i S_e^j \nabla_i \nabla_j \frac{1}{r} - \vec{S}_p \vec{S}_e \nabla^2 \frac{1}{r} \Big]$$

— introduce $T_{ij} = (\nabla_i \nabla_j - \delta_{ij} \nabla^2) \frac{1}{r}$

$$H' = \frac{g g_p \alpha}{4 m_e M_p} \sum_{ij} S_p^i S_e^j T_{ij}(r) \qquad (*)$$

— Discuss $T_{ij}(r)$

$$T_{ij}(r) = \left(\nabla_i \nabla_j - \delta_{ij} \nabla^2 \right) \frac{1}{r} \underbrace{- \frac{1}{3} \delta_{ij} \sum_k T_{kk} + \frac{1}{3} \delta_{ij} \sum_k T_{kk}}_{} = //$$

$$\sum_k T_{kk} = \sum_k \left(\nabla_k^2 - \delta_{kk} \nabla^2 \right) \frac{1}{r} = \left(\nabla^2 - 3 \nabla^2 \right) \frac{1}{r} = -2 \nabla^2 \frac{1}{r}$$

$$// = \quad T_{ij} = T_{ij} - \frac{1}{3} \delta_{ij} \sum_k T_{kk} + \frac{1}{3} \delta_{ij} \sum_k T_{kT} = //$$

$$= // \left(\nabla_i \nabla_j - \delta_{ij} \nabla^2 \right) \frac{1}{r} + \frac{2}{3} \delta_{ij} \nabla^2 \frac{1}{r} - \frac{2}{3} \delta_{ij} \nabla^2 \frac{1}{r}$$

$$= \left(\nabla_i \nabla_j - \frac{1}{3} \delta_{ij} \nabla^2 \right) \frac{1}{r} - \frac{2}{3} \delta_{ij} \nabla^2 \frac{1}{r}$$

$$T_{ij} = \left(\nabla_i \nabla_j - \frac{1}{3} \delta_{ij} \nabla^2 \right) \frac{1}{r} - \frac{2}{3} \delta_{ij} \nabla^2 \frac{1}{r}$$

$$\nabla^2 \frac{1}{r} = -4\pi \delta(r)$$

$$T_{ij} = \left(\nabla_i \nabla_j - \frac{1}{3}\delta_{ij}\nabla^2\right)\frac{1}{r} + \frac{2}{3}4\pi\,\delta_{ij}\,\delta(r)$$

Remember (*)

Dipole

$$H' = \frac{g g_p \alpha}{4mM} \sum_{ij} S_p^i S_e^j \left(\nabla_i \nabla_j - \frac{1}{3}\delta_{ij}\nabla^2\right)\frac{1}{r} +$$

Contact

$$+ \frac{g g_p \alpha}{4mM} \cdot \frac{8\pi}{3}\, \vec{S}_p \cdot \vec{S}_e\, \delta(r)$$

$$H'_{Dipole} = \frac{g g_p \alpha}{4mM} \sum_{ij} S_p^i S_e^j \left(\nabla_i \nabla_j - \frac{1}{3}\delta_{ij}\nabla^2\right)\frac{1}{r} =$$

$$= \frac{g g_p \alpha}{4mM} \frac{1}{r^5} \sum_{ij} S_p^i S_e^j \left(3 r_i r_j - \delta_{ij} r^2\right) ?$$

$$H'_{cont} = \frac{g g_p \alpha}{4mM} \frac{8\pi}{3} (S_e \cdot S_p)\, \delta^3 r$$

- Consider Hydrogen Ground state
  $\ell = 0$, S state

- show That $\langle \phi^{10}_{\frac{1}{2}m} | H'_{Dipole} | \phi^{10}_{\frac{1}{2}m} \rangle = 0$
  HW

- $\Delta'_{cont} = \langle \phi^{10P}_{\frac{1}{2}m} | H'_{contact} | \phi^{10P}_{\frac{1}{2}} \rangle =$

using $I = \int |r\rangle\langle r| d^3 r$

$$= \int \langle \phi^{10,P}_{\frac{1}{2}m} | r \rangle | H'_{cont} | \langle r | \phi^{10}_{12} \rangle\, d^3 r$$

- Remember for $\ell = 0$ $\Phi_{S m_S} = \Psi_{S, m_S} = R_{10}(r) Y_{00} |S_e\rangle |S_p\rangle$

$- \text{where} \quad Y_{00} = \frac{1}{\sqrt{4\pi}}$ $\qquad R_{10} = \frac{2}{a^{3/2}} e^{-r/a}$

$$a_B = \frac{\hbar}{\alpha m c} = \frac{1}{\alpha m}$$

$$- H'_{cont} = \frac{g g_p \alpha}{4 m M_p} \frac{8\pi}{3} \langle S_e S_p | \vec{S} \cdot \vec{P} | \vec{S_e} \vec{S_p} \rangle \cdot \int \delta(0) |\psi^{10}_{\frac{1}{2} m}(r)|^2 d^3 r$$

$$= \frac{g g_p \alpha}{4 m M_p} \frac{8\pi}{3} \cdot \frac{1}{\pi} m^3 \alpha^3 \langle S_e S_p | \vec{S_e} \vec{S_p} | S_e S_p \rangle$$

$$= \frac{g g_p \alpha^4 m^2}{3 M_p} \langle | 2 S_p S_e | \rangle$$

$\longleftarrow \quad \vec{F} = S_p + S_e + L$

For $L = 0$

$\vec{F} = \vec{S_p} + \vec{S_e} \quad \Big| \quad F^2 = S_p^2 + S_e^2 + 2 \vec{S_p} \vec{S_e}$

$$2 S_p S_e = F^2 - S_p^2 - S_e^2$$

$$H'_{cont} = \frac{g g_p \alpha^4 m^2}{3 M_p} \underbrace{\langle | F^2 - S_p^2 - S_e^2 | \rangle}_{\text{Term}}$$

if $F = 1 \qquad \text{Term} = 2 - \frac{3}{2} = \frac{1}{2}$

$\qquad\qquad\qquad\qquad \frac{4}{2} - \frac{3}{2}$

if $F = 0 \qquad \text{Term} = -\frac{3}{2}$

$$\Delta_{cont} = \Delta_{HFS} = \frac{\alpha^4}{3}\frac{m^2}{M_p} g_p \begin{pmatrix} \frac{1}{2} \\ -\frac{3}{2} \end{pmatrix} \quad F \begin{matrix} \nearrow 1 \\ \searrow 0 \end{matrix}$$

— Splitting

$$\Delta^{F=1} - \Delta^{F=0} = \frac{\alpha^4}{3}\frac{m^2}{M_p} g_p \left(\frac{1}{2} + \frac{3}{2}\right) =$$

$$= \frac{2}{3}\alpha^4 \cdot \frac{m^2}{M_p} g_p = \frac{2}{3}\frac{1}{(137)^4}\frac{(0.51)^2}{938.272} \cdot 2 \cdot 2.278 =$$

$$= 5.85008 \times 10^{-6} eV$$

— Transition $\lambda = \frac{2\pi}{\Delta_{split}} = \frac{2\pi}{5.85 \times 10^{-6} eV} = 21.2 cm$

— Discovered in 1951