L14

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General case of ℓ and $s=\frac{1}{2}$

- $\ell=1,\ s=\frac{1}{2}$ → $j=\frac{3}{2}$, $j=\frac{1}{2}$

- $\ell,\ s=\frac{1}{2}$ → $j=\ell+\frac{1}{2}$, $j=\ell-\frac{1}{2}$ $\qquad |\psi\rangle=\sum_n x_n|x_n\rangle$

$$|\phi_{j\,m_j}\rangle=\sum_{m_\ell,m_s}C_{m_\ell,m_s}|\psi_{\ell m_\ell,\,s m_s}\rangle$$

$m_j=m_\ell+m_s$, $\frac{1}{2}\ \ -\frac{1}{2}$

$m_\ell=m_j-m_s$

if $m_s=\frac{1}{2}$: $m_\ell=m_j-\frac{1}{2}$

if $m_s=-\frac{1}{2}$: $m_\ell=m_j+\frac{1}{2}$

$$|\phi_{\ell+\frac{1}{2},m_j}\rangle=C_{m_j-\frac{1}{2}}|\psi_{\ell\,m_j-\frac{1}{2},\,s\frac{1}{2}}\rangle+C_{m_j+\frac{1}{2}}|\psi_{\ell\,m_j+\frac{1}{2},\,s-\frac{1}{2}}\rangle$$

$m_j=\ell+\frac{1}{2}$ max, m_ℓ max, m_s max

$$|\phi_{\ell+\frac{1}{2},\ell+\frac{1}{2}}\rangle=|\psi_{\ell,\ell,\,s\frac{1}{2}}\rangle$$

Top state $\qquad$ Top

$m_j=m$

$C_{m_j-\frac{1}{2}}=\alpha_m$

$C_{m_j+\frac{1}{2}}=\beta_m$

$$\alpha_{m^{max}}=\alpha_{\ell+\frac{1}{2}}=1,\qquad \beta_{m^{max}}=0$$

$$|\phi_{\ell+\frac{1}{2},m}\rangle=\alpha_m|\psi_{\ell\,m-\frac{1}{2},\,s\frac{1}{2}}\rangle+\beta_m|\psi_{\ell\,m+\frac{1}{2},\,s-\frac{1}{2}}\rangle$$

$m\to m-1$

J_-

$$\sqrt{(j+m)(j-m+1)}\ \text{or}\ \sqrt{s(s+1)-m(m-1)}\ |\phi_{\ell+\frac{1}{2},m-1}\rangle=\sqrt{(j+m)(j-m+1)}\left(\alpha_{m-1}|\psi_{\ell,m-\frac{3}{2},\,s\frac{1}{2}}\rangle+\beta_{m-1}|\psi_{\ell,m-\frac{1}{2},\,s-\frac{1}{2}}\rangle\right)\quad (a)$$

$$(L_-+S_-)\left[\alpha_m|\psi_{\ell\,m-\frac{1}{2},\,s\frac{1}{2}}\rangle+\beta_m|\psi_{\ell,m+\frac{1}{2},\,s-\frac{1}{2}}\rangle\right]=$$

$$=\alpha_m L_-|\psi_{\ell m-\frac{1}{2},\,s\frac{1}{2}}\rangle+\beta_m L_-|\psi_{\ell,m+\frac{1}{2},\,s-\frac{1}{2}}\rangle+$$
$$\alpha_m S_-|\psi_{\ell m-\frac{1}{2},\,s\frac{1}{2}}\rangle+\beta_m S_-|\psi_{\ell,m+\frac{1}{2},\,s-\frac{1}{2}}\rangle=0$$

$$=\alpha_m\sqrt{\ell(\ell+1)-(m-\tfrac{1}{2})(m-\tfrac{1}{2}-1)}\,|\psi_{\ell m-\frac{3}{2},\,s\frac{1}{2}}\rangle+\beta_m\sqrt{\ell(\ell+1)-(m+\tfrac{1}{2})(m-\tfrac{1}{2})}\,|\psi_{\ell,m-\frac{1}{2},\,s-\frac{1}{2}}\rangle$$

$\sqrt{(\ell+m-\frac{1}{2})(\ell-m+\frac{3}{2})}$ $\qquad$ $\sqrt{(\ell+m+\frac{1}{2})(\ell-m+\frac{1}{2})}$

$$\alpha_m\sqrt{s(s+1)-\tfrac{1}{2}(\tfrac{1}{2}-1)}\,|\psi_{\ell\,m-\frac{1}{2},\,s-\frac{1}{2}}\rangle$$

$(\frac{3}{4}+\frac{1}{4})=1$

$$=\alpha_m\sqrt{(\ell+m-\tfrac{1}{2})(\ell-m+\tfrac{3}{2})}\,|\psi_{\ell,m-\frac{3}{2},\,s\frac{1}{2}}\rangle+\beta_m\sqrt{(\ell+m+\tfrac{1}{2})(\ell-m+\tfrac{1}{2})}\,|\psi_{\ell m-\frac{1}{2},\,s\frac{1}{2}}\rangle$$
$$+\ \alpha_m|\psi_{\ell m-\frac{1}{2},\,s-\frac{1}{2}}\rangle\quad (b)$$

$$\alpha_{m-1}\sqrt{(j+m)(j-m+1)}=\alpha_m\sqrt{(\ell+m-\tfrac{1}{2})(\ell-m+\tfrac{3}{2})}$$

$j=\ell+\frac{1}{2}$

$$\alpha_{m-1}\sqrt{(\ell+m+\tfrac{1}{2})(\cancel{\ell-m+\tfrac{3}{2}})}=\alpha_m\sqrt{(\ell+m-\tfrac{1}{2})(\cancel{\ell-m+\tfrac{3}{2}})}$$

$-j\le m'\le j$

$m'^{\,max}=\ell+\frac{1}{2}$

$$\frac{\alpha_m}{\sqrt{\ell+m+\frac{1}{2}}}=\frac{\alpha_{m-1}}{\sqrt{(\ell+m-\frac{1}{2})}}=\frac{\alpha_{m-1}}{\sqrt{(\ell+(m-1)+\frac{1}{2}}}=\frac{\alpha_{m'}}{\sqrt{\ell+m'+\frac{1}{2}}}=\frac{\alpha_{\ell+\frac{1}{2}}}{\sqrt{\ell+\ell+\frac{1}{2}+\frac{1}{2}}}=\ldots$$

$$\frac{\alpha_m}{\sqrt{\ell+m+\frac{1}{2}}}=\frac{1}{\sqrt{2\ell+1}}\ \longrightarrow\ \boxed{\alpha_m=\sqrt{\frac{\ell+m+\frac{1}{2}}{2\ell+1}}}$$

$J^3_m = \qquad L_m + J_w = 1$

$$\vec{J} = \vec{L} + \vec{S}$$

$$\begin{cases} j = \ell + \frac{1}{2} \\ j = \ell - \frac{1}{2} \end{cases}$$

$$\vec{J} = \vec{J}_1 + \vec{J}_2$$

$j = j_1 + j_2$

$j = j_1 + j_2 - 1$

$j = j_1 + j_2 - 2$

$\ldots$

$j = |j_1 - j_2|$

$m_j = m_{j1} + m_{j2}$

Example

$S_1 \quad S_2$

$$S = S_1 + S_2$$

$S = 1 \quad m_s = 1, 0, -1$ } triplet state

$S = 0 \longrightarrow m_s = 0$ } $\to$ singlet

$$m_s = m_{s_1} + m_{s_2}$$

$m_s = 1, 0, 0, -1$

$S = 1 \quad m_s =$

$S = 0 \quad m_s =$

$d = 1 \quad (p, n)$

$p - n \quad L = 0, 2$

$L = 1$

$S = \frac{1}{2} \quad p = \uparrow u \downarrow u \uparrow d$

$\frac{1}{2} \quad n = \uparrow d \downarrow d \uparrow u$

$\frac{3}{2} \quad \Delta^{++} \to \uparrow u_R \uparrow u_G \uparrow$

$\Rightarrow$ Recursion Formula

$$\vec{J} = \vec{J}_1 + \vec{J}_2$$

$|\Phi_{\ldots}\rangle \qquad |\Psi_{j_1 m_1, j_2 m_2}\rangle$

$|\phi_{jm_j}\rangle$

$$\langle \psi_{j_1 m_1 j_2 m_2} | J_\pm | \phi_{j m_j}\rangle$$

$$J_\pm |\phi_{jm_j}\rangle = \sqrt{j(j+1) - m_j(m_j\pm1)}\, |\phi_{j m_j\pm1}\rangle$$

$$\langle \psi_{j_1m_1j_2m_2}|J_\pm|\phi_{jm_j}\rangle = \sqrt{j(j+1)-m_j(m_j\pm1)}\,\langle\psi_{j_1m_1j_2m_2}|\phi_{jm_j\pm1}\rangle$$

$$\langle\psi_{j_1m_1j_2m_2}| J_{1\pm}+J_{2\pm}|\phi_{jm_j}\rangle$$

$$(J_{1\mp}+J_{2\mp})|\psi_{j_1m_1j_2m_2}\rangle = \sqrt{j_1(j_1+1)-m_{j_1}(m_{j_1}\mp1)}\,|\psi_{j_1m_{j_1}\mp1\,j_2m_2}\rangle + \sqrt{j_2(j_2+1)-m_{j_2}(m_{j_2}\mp1)}\,|\psi_{j_1m_{j_1}\,j_2m_2\mp1}\rangle$$

$$\boxed{\sqrt{j_1(j_1+1)-m_1(m_1\mp1)}\,\underbrace{\langle\psi_{j_1m_1\mp1,\,j_2m_2}|\phi_{jm}\rangle}_{C^{jm}_{j_1m_1\mp1;\,j_2m_2}} + \sqrt{j_2(j_2+1)-m_{j_2}(m_{j_2}\mp1)}\,\underbrace{\langle\psi_{j_1m_1,\,j_2m_2\mp1}|\phi_{jm}\rangle}_{C^{jm}_{j_1m_1;\,j_2m_2\mp1}} = \sqrt{j(j+1)-m_j(m_j\pm1)}\,\underbrace{\langle\psi_{j_1m_1j_2m_2}|\phi_{jm_j\pm1}\rangle}_{C^{jm\pm1}_{j_1m_1;\,j_2m_2}}}$$

$$|\phi_{jm_j}\rangle = \sum_{\substack{m_{j1}\\ m_{j2}}} C^{jm_j}_{j_1m_1;\,j_2m_2}\,|\psi_{j_1m_1,\,j_2m_2}\rangle$$

$|\psi\rangle = \sum\alpha$

$$\boxed{C^{jm_j}_{j_1m_1;\,j_2m_2} = \langle\psi_{j_1m_1;\,j_2m_2}|\phi_{jm_j}\rangle}$$

Clebsch – Gordan Coefficients

$m_j = j$

$$C^{jj}_{j_1j_1;\,j_2j_2} = 1$$

m_1

$M_1 = J_1$ J_2

$J_1 = \frac{3}{2}$ $J_2 = 1$