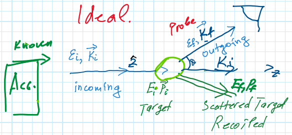
L17

22-March-2021

Potential Scattering

Ideal.



Kinematics:
(Energy - Momentum Conservation)

$\mathcal{E}_i + E_i = \mathcal{E}_f + E_f$ – Energy

$\vec{K}_i + \vec{P}_i = \vec{K}_f + \vec{P}_f$ $\begin{Bmatrix} x \\ y \\ z \end{Bmatrix}$ – Momentum

$\vec{P}_i = 0$

Natural Units $\hbar = 1$ $c = 1$ $E_{Rest} = mc^2 = m$

Conditions

① Non Relativistic

$$\begin{cases} \mathcal{E}_i = m_i + \frac{K_i^2}{2m_i} \\ \mathcal{E}_f = m_f + \frac{K_f^2}{2m} \\ E_i = M_i + \frac{P_i^2}{2M_i} = M_i \\ E_f = M_f + \frac{P_f^2}{2M_f} \end{cases}$$

Energy Momentum Conservation

NON Relativistic
$$\begin{cases} \boxed{m_i + \frac{K_i^2}{2m_i}} + \boxed{M_i} = m_f + \frac{K_f^2}{2m_f} + \boxed{M_f + \frac{P_f^2}{2M_f}} \\ \vec{K}_i = \vec{K}_f + \vec{P}_f \end{cases}$$

Elastic
$$\begin{cases} m + \frac{K_i^2}{2m} + M = m + \frac{K_f^2}{2m} + M + \frac{P_f^2}{2M} \\ \vec{K}_i = \vec{K}_f + \vec{P}_f \end{cases}$$

② Considering in the Lab Frame
$\boxed{\vec{P}_i = 0}$ (target is initially at rest)

③ Elastic Scattering

$$\left.\begin{matrix} m_i = m_f = m \\ M_i = M_f = M \end{matrix}\right\}$$

④ Target is infinitely heavy

$$\boxed{\frac{P_f^2}{2M_f} \to 0} \implies \begin{cases} \frac{K_i^2}{2m} = \frac{K_f^2}{2m} \\ \vec{K}_i = \vec{K}_f + \vec{P}_f \end{cases}$$

Energy Momentum Conservation

$|K_i| = |K_f| \Leftarrow \equiv K$

$\vec{P}_f = \vec{K}_i - \vec{K}_f$ ($\vec{q}$ – momentum transfer)

$\vec{K}_i \cdot \vec{K}_f = K_i K_f \cos\theta$

$\vec{q} = \vec{K}_i - \vec{K}_f \neq 0$

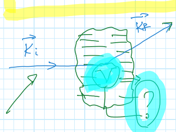
$$|q| = \sqrt{(\vec{K}_i - \vec{K}_f)^2} = \sqrt{K_i^2 - 2\vec{K}_i \cdot \vec{K}_f + K_f^2} =$$
$$= \sqrt{K_i^2 - 2K_i K_f \cos\theta + K_f^2} = \sqrt{2K^2 - 2K^2\cos\theta} =$$
$$= K\sqrt{2(1-\cos\theta)} = K\sqrt{2\left(1 - 1 + 2\sin^2\frac{\theta}{2}\right)} =$$

$$\cos\theta = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} =$$
$$= 1 - 2\sin^2\frac{\theta}{2}$$

$$|| = K\sqrt{4\sin^2\frac{\theta}{2}} = \boxed{2K\sin\frac{\theta}{2}}$$

Targets Participation in the Scattering is providing Force (Interaction) Field in which incoming Particle Scatters

Potential Scattering



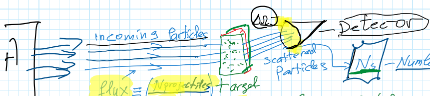
$|K_i| = |K_f| = K$

$\vec{q} = \vec{K}_i - \vec{K}_f$

$\vec{P}_f = \vec{q}$

$\Rightarrow$ **Cross Section**

(what is being measured experimentaly?)



N_s – Number of scattered events per solid angle

flux $\equiv \left(\frac{N_{projectiles}}{cm^2 \cdot Sec}\right)$ – Luminosity 10^{32}

N_T – Number of Particles in the target 10^{26}

$\boxed{N_s^{Total}} = N_s \cdot \Delta\Omega$ (Per Second)

$$N_s^{Total} \sim \text{flux} \cdot N_T = \frac{N_{proj}}{cm^2 \; Sec} \cdot N_T$$

$\frac{Number}{Sec}$ $\left[\frac{N_s^{Total}}{s}\right]$ $\frac{1}{Sec.}$ $\frac{1}{cm^2 \cdot Sec.}$

$$N_s \boxed{\Delta\Omega} = \boxed{\Delta\sigma} \cdot \frac{N_{proj}}{cm^2\,Sec} \cdot N_T$$

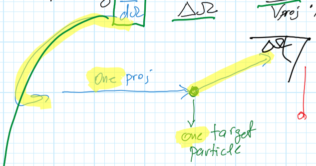
– Units

$$\boxed{\frac{1}{sec}} = [\Delta\sigma] \boxed{\frac{1}{cm^2\,Sec}}$$

$[\Delta\sigma] = \boxed{cm^2}$ – $\boxed{\text{area}} \Rightarrow$ **Cross Sections**

– Meaning
$$\boxed{\frac{d\sigma}{d\Omega}} = \frac{\Delta\sigma}{\Delta\Omega} = \frac{N_s}{V_{proj} \cdot N_T}$$

cross sectional area for one projectile to scatter off one one target particle to one unit of solid angle of the detector



$\frac{d\sigma}{d\Omega}$ – differential Cross Section

$$\boxed{\sigma_{TOT} =} \int \frac{d\sigma}{d\Omega} \cdot d\Omega \text{ – } \boxed{\text{Total Cross section}}$$

⟹ Relating it to Quantum Mechanics

$\rho(r) = |\psi^*(r)\psi(r)|$

$\vec{j}$ = probability current

One projectile, proj, $j_{incoming}$, j_{in}, $j_{outgoing}$, j_{out}

one target particle as a Source of potential

One target ⟹ $V(r)$ — potential Energy

– Remember $\vec{j}$ — probability current density

Schroedinger Wave Equation

$$i\hbar \frac{\partial}{\partial t}\psi(\vec r,t) = -\frac{\hbar^2 \nabla^2}{2m}\psi(\vec r,t) + V(r)\psi(\vec r,t)$$

$V(r) = 0$

$$\vec{j} = -\frac{i}{2m}\left(\psi^*(\vec r,t)\vec\nabla\psi(\vec r,t) - \psi(\vec r,t)\vec\nabla\psi^*(\vec r,t)\right)$$

– Satisfying Continuity Equation

$$\boxed{\frac{\partial \rho}{\partial t} = -\vec\nabla \vec{j}} \qquad \rho = \psi^*(\vec r,t)\psi(\vec r,t)$$

– To describe the Scattering in QM we associate
$\vec{j}_{in}$ – with the incoming particle
$\vec{j}_{out}$ – with the scattered particle
$V(r)$ – potential with the target.

$\frac{d\sigma^{QM}}{d\Omega}$

⟹ Quantum Mechanical Scattering

$$d\sigma^{QM} = \frac{\vec{j}_{out} \text{ into the Detector}}{j_{inc}^2}$$

$$d\sigma^{QM} = \frac{j_{out}\, d\vec{s}}{j_{inc}^2}$$

$d\vec s$ – cross sect Area of my detector, cm^2

$[d\sigma^{QM}] = $ area

$$d\sigma^{QM} = \frac{\vec{j}_{out}\cdot d\vec{s}}{j_{inc}^2}$$

① Consider first the Potential $V(r)$

Since the target hase a finite size
In QM we can associate this with the condition

$$V(r)\Big|_{r\to\infty} \longrightarrow 0$$

ⓐ simple approximation: there is such a R
$V(r) = V(r)$, $r \le R$
$V(r) = 0$, $r > R$ ①

ⓑ More sofisticated approximation
$r^{1+\varepsilon} V(r)\Big|_{r\to\infty} \longrightarrow 0$, with $\varepsilon > 0$

② Probability Density Current for incoming Particle

Because of condition ① $V(r)\big|_{r\to\infty} \to 0$
we can consider the incoming particle to be free
with momentum $\vec{k}_i$, Energy $E = \frac{k_i^2}{2m} = \frac{k^2}{2m}$
$\|$
$|\vec{k}|$ (remember $|\vec{k}_i| = |\vec{k}| = k$)

– The Wave function of free particle

$$\psi^{inc}_{free}(\vec r,t) \equiv \phi(\vec r,t) = \frac{1}{(2\pi)^{3/2}} e^{i\vec{k}\cdot\vec{r}}\, e^{-iE_i t} = \phi(r)\, e^{-iEt}$$

$$\rho = \phi^*(\vec r,t)\phi(\vec r,t) = \boxed{\frac{1}{(2\pi)^3}} = \phi^*(r)\phi(r) \qquad \text{Sch. Eq.} \quad \boxed{(\nabla^2 + k^2)\phi(r) = 0} \text{ HW}$$

$$\vec{j}_{inc} = -\frac{i}{2m}\left(\phi^*(\vec r,t)\vec\nabla\phi - \phi\vec\nabla\phi^*\right) = //$$

$$\vec\nabla\phi = i\vec{k}\phi, \quad \vec\nabla\phi^* = -i\vec{k}$$

$$// = -\frac{i}{2m}\left(i\vec{k}\phi^*\phi + i\vec{k}\phi\phi^*\right) = \frac{2\vec{k}}{2m}\frac{1}{(2\pi)^3} = \frac{\vec{k}}{m}\frac{1}{(2\pi)^3}$$

$$\boxed{\vec{j}_{inc} = \frac{\vec{k}}{m}\frac{1}{(2\pi)^3}} \qquad (\vec{k})^{\hat z} = K \qquad \frac{K}{m} = V \qquad \boxed{j_{inc,z} = V\frac{1}{(2\pi)^3}} \qquad \boxed{V^z = |V|}$$

⟹ Calculation of $\vec{j}_{out}$

(we are interested in j_{out} far away from the target, (entering to the detector))

– Scattered particle passed through the Potential field

- To find the wave function of this particle

Consider complete Schroedinger Equation (for stationary $V(r)$)

- stationary state $\boxed{\Psi(r,t) = e^{-i\frac{\varepsilon_f t}{\hbar}} \Psi(r)}$ stationary Solution

$$\boxed{\left(-\frac{1}{2m}\nabla^2 + V(r)\right)\Psi(r) = \varepsilon_f \Psi(r)} \quad (HW)$$

$\varepsilon_f > 0$

- introduce $\boxed{U(r) = 2mV(r)}$ $\quad \varepsilon_f = \frac{k_f^2}{2m}$ $\quad \boxed{k_f = k_i = k}$

$k_f^2 = 2m\varepsilon_f = k^2$

$$\boxed{(\nabla^2 + k^2)\Psi(r) = U(r)\Psi(r)} \quad (*)$$

- in case $\boxed{U(r) = 0}$ $\quad \Psi(r) = \phi(r) = \frac{1}{(2\pi)^{3/2}} e^{i\vec{k}\vec{r}}$

- Introduce $\Psi(r) = \phi(r) + \chi(r)$ $\quad [\nabla^2 + k^2]\phi(r) = 0$

$\phi(r) = \frac{1}{(2\pi)^{3/2}} e^{i\vec{k}\vec{r}}$ $\quad$ Scattered wave function

- $\chi(r)$ – scattering part of the wave function

$$\boxed{\Psi(r) = \frac{1}{(2\pi)^{3/2}} e^{i\vec{k}\vec{r}} + \chi(r)} \quad (+)$$

- We also know that when $r \to \infty$ $U(r) \to 0$ $\quad \boxed{(\nabla^2 + k^2)\phi(r) = 0}$

$\Psi(r) \to \phi(r)$

- Consider the Schroedinger Equation

$$(\nabla^2 + k^2)\Psi(r) = U(r)\Psi(r) \Rightarrow$$

using $[\nabla^2 + k^2]\phi(r) = 0$ for all r

$$\Rightarrow (\nabla^2 + k^2)[\phi(r) + \chi(r)] = U(r)\Psi(r) = //$$

$$// \boxed{(\nabla^2 + k^2)\chi(r) = U(r)\Psi(r)}$$

$\nabla^2 \to$ polar coordinates

- consider limit $r \to \infty$ $U(r) = 0$

$r \to \infty$

- Consider in polar coordinates

$$\boxed{\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2} + k^2\right]\chi(r) = 0}$$

$r \to \infty$

$$\boxed{\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + k^2\right]\chi(r)\Big|_{r\to\infty} = 0}$$

- introduce $\boxed{\chi(r) = \frac{A(r,\theta,\varphi)}{r}}$

$$\frac{\partial}{\partial r}\frac{A(r,\theta,\varphi)}{r} = -\frac{A}{r^2} + \frac{A'}{r}$$

$$r^2\frac{\partial}{\partial r}\frac{A}{r} = -A + rA' \Rightarrow \frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}A = -A' + A' + rA''$$

$$\boxed{\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}A = \frac{A''}{r}}$$

- Eq(1) becomes

$$\boxed{\left(\frac{A''}{r} + \frac{k^2}{r}A\right) = 0} \quad A'' + k^2 A = 0 \qquad A'' = \frac{\partial^2 A}{\partial r^2}$$

- Substitude $\boxed{A(r,\theta,\varphi) = \frac{f(\theta,\varphi)}{(2\pi)^{3/2}} e^{\pm ikr}}$ not a dot product

Solutions

$$-k^2A + k^2A = 0$$

- Therefor the solution for $\chi(r)$ is $\quad r \to \infty$

$$\boxed{\chi_\pm(r) = \frac{f(\theta,\varphi)}{(2\pi)^{3/2}}\frac{e^{\pm ikr}}{r}}$$

- $\boxed{(+) \text{ – diverging spherical wave}}$

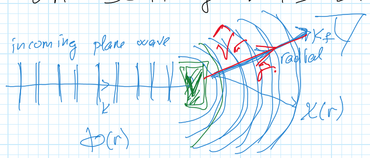
- (–) – converging spherical wave

not relevant

- For Scattering state we choose

$$\boxed{\chi_+(r) = \frac{f(\theta,\varphi)}{(2\pi)^{3/2}} \frac{e^{ikr}}{r}} \quad kr$$

- QM Scattering Looks Like



- to calculate $\vec{J}_{out}$ we should calculate it for $\chi(r)$ $\boxed{\vec{J}_{out}}$

$$\boxed{\vec{J}_{out} = -\frac{i}{2m}\left(\chi^* \vec{\nabla} \chi - \chi \vec{\nabla} \chi^*\right) =}$$

$$= -\frac{i}{2m}\left(\frac{|f(\theta,\varphi)|^2}{(2\pi)^3} \frac{e^{-ikr}}{r} \vec{\nabla} \frac{e^{ikr}}{r} - \frac{|f(\theta,\varphi)|^2}{(2\pi)^3} \frac{e^{ikr}}{r} \vec{\nabla} \frac{e^{-ikr}}{r}\right) = //$$

- $\vec{\nabla} \frac{e^{ikr}}{r} = \left[\vec{\nabla} e^{ikr} \cdot \frac{1}{r} + \nabla \frac{1}{r} e^{ikr}\right]$

$$\vec{\nabla} e^{ikr} = ik e^{ikr} \vec{\nabla} \cdot r = ik e^{ikr} \frac{\partial}{\partial r_i} \sqrt{x^2+y^2+z^2} = ik e^{ikr} \frac{\vec{r}}{r}$$

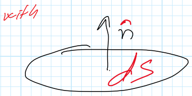
$$\nabla \frac{1}{r} = \frac{\partial}{\partial r_i} \frac{1}{\sqrt{x^2+y^2+z^2}} = -\frac{\vec{r}}{r^3}$$

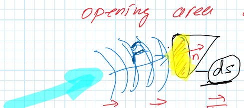
$$// = -\frac{i}{2m} \frac{|f(\theta,\varphi)|^2}{(2\pi)^3} \left(\frac{1}{r}\left(ik\frac{\vec{r}}{r^2} - \frac{\vec{r}}{r^3}\right) - \frac{1}{r}\left(-ik\frac{\vec{r}}{r^2} - \frac{\vec{r}}{r^3}\right)\right)$$

$$= \frac{2k}{2m} \frac{|f(\theta,\varphi)|^2}{(2\pi)^3 r^2} \frac{\vec{r}}{r} = \frac{|f(\theta,\varphi)|^2}{(2\pi)^3 r^2} v_r$$

$$\boxed{J_{out} = \frac{|f(\theta,\varphi)|^2}{(2\pi)^3 r^2} v_r}$$

- Quantum particles entering detector with opening area dS





$$\vec{J}_{out} \cdot d\vec{S} = \vec{J}_{out} \cdot \vec{n} \cdot dS = \frac{|f(\theta,\varphi)|^2}{(2\pi)^3 r^2} \vec{v}_r \cdot \vec{n} \cdot dS = \quad \text{since } \hat{n} \parallel \hat{r}$$

$$\vec{J}_{out} \cdot d\vec{S} = \frac{|f(\theta,\varphi)|^2 |V|}{(2\pi)^3 r^2} dS = \frac{|f(\theta,\varphi)|^2 |V| r^2 d\Omega}{(2\pi)^3 r^2} = \frac{|f(\theta,\varphi)|^2}{(2\pi)^3} |V| \cdot d\Omega.$$

$$dS = r^2 d\Omega$$

- incoming Quantum particle along z-axis

$$J_{inc,z} = \frac{V_z}{(2\pi)^3} = \frac{k_z}{m(2\pi)^3} \Rightarrow \frac{k}{m(2\pi)^3} = \frac{V}{(2\pi)^3}$$

if the experiment is set-up such way that $\vec{k}_i = k_z$ — along z

- QM Differential Cross Section

$$d\sigma = \frac{\vec{J}_{out} \cdot d\vec{S}}{J_{inc}} = \frac{\frac{|f(\theta,\varphi)|^2}{(2\pi)^3} V d\Omega}{V/(2\pi)^3} = |f(\theta,\varphi)|^2 \cdot d\Omega$$

$$\boxed{\frac{d\sigma}{d\Omega}^{QM} = |f(\theta,\varphi)|^2} = \frac{d\sigma}{d\Omega}^{exp}$$

measured

$$\boxed{\sigma_{TOT} = \int \frac{d\sigma}{d\Omega} d\Omega = \int |f(\theta,\varphi)|^2 d\Omega}$$

- $f(\theta,\varphi)$ — scattering Amplitude

$\boxed{V(r)}$ → Structure of the target

? ?