L18

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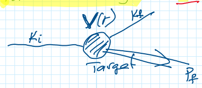
Scattering Amplitude

- In the Last Lecture $E_i = M_i$ $\quad E = m + \frac{p^2}{2m}$ $\quad m_i = m_f$, $M_i = M_f$
- Kinematics of Scattering: Lab Frame, Non Relativistic, Elastic, Heavy Target, Potential scattering $\quad \frac{P_f^2}{2M} \sim 0$, $M \to \infty$

$|K_i| = |K_{f}| = K$

$\vec{q} = \vec{K_i} - \vec{K_f} = \vec{P_f}$

momentum transfer.



Differential and Total Cross Sections of Scattering

- experimental definition

luminosity: area · st

$$\left[\frac{d\sigma^{exp}}{d\Omega}\right] = \frac{N_{sc}}{N_{proj} \, N_T}$$

$N_{proj} \sim 10^{32}$, $N_T \sim 10^{26}$

Diff. Cross Section is for
- one projectile to scatter off
- one target particle
- to one unit of solid angle of the detector

one proj, free. one target particle. free

$$\boxed{\sigma_{TOT}^{exp} = \int \frac{d\sigma^{exp}}{d\Omega} \cdot d\Omega}$$

- Quantum Mechanical Definition

⇒ ① Identifying projectile particles with probability density current $\vec{J}$

$$\vec{J} = \frac{-i}{2m}\left(\psi^*(r,t)\vec{\nabla}\psi(r,t) - \psi(r,t)\vec{\nabla}\psi^*(r,t)\right)$$

⇒ ② Identifying target particle with stationary potential $V(r)$ $\quad \psi \to \phi_k(r)$

⇒ ③ Then
- incoming particle is described by $\vec{J}_{inc}$
- Scattered/outgoing particle is described by $\vec{J}_{out}$
- Target particle described by $V(r)$
- Finite size of the target is described by the condition $V(r)\big|_{r\to\infty} \to 0$
  - Example $V(r) = \begin{cases} V(r): & r \le R \\ 0 & r > R \end{cases}$

⇒ QM Differential Cross Section is defined

$$d\sigma^{QM} = \frac{\vec{J}_{out} \cdot d\vec{s}}{J_{in,z}} \qquad d\vec{s} = ds\,\hat{n}$$

$J_{in,z}$ → [V] → J_{out}

- Calculation of Diff. Cross Section in QM

- J_{inc}, particle is described by free wave function

$$\psi_{inc}^{free}(r,t) = \phi(r)\, e^{-i\varepsilon t} \qquad \phi(r) = \frac{1}{(2\pi)^{3/2}} e^{i\vec{k}\cdot\vec{r}}$$

where $(\nabla^2 + k^2)\phi(r) = 0$

$$\boxed{\vec{J}_{inc} = \frac{\vec{k}}{m}\frac{1}{(2\pi)^3} = \frac{\vec{v}}{(2\pi)^3}}$$

- $\vec{J}_{out}$: $\quad \psi(r,t) = e^{-i\varepsilon_f t}\,\psi(r)$ $\qquad V(r)$

(*) $\quad (\nabla^2 + k^2)\psi(r) = U(r)\psi(r)$ $\qquad$ ⇒ $U(r) = 2mV(r)$, $\quad k_f^2 = 2m\varepsilon_f = k^2$, $\quad |K_i| = |K_f| = K$

to solve this equation

- expressed $\psi(r) = \phi_k(r) + \chi(r) = \frac{1}{(2\pi)^{3/2}} e^{i\vec{k}\cdot\vec{r}} + \chi(r)$ $\qquad (\nabla^2 + k^2)\phi(r) = 0$

Polar coordinates and obtained

$$\boxed{(\nabla^2 + k^2)\chi(r) = U(r)\psi(r)}$$

- considered in polar coordinates in the domain of $r \to \infty$ $\qquad U(r)\big|_{r\to\infty} \to 0$

Polar coordinates

$$\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + k^2\right]\chi(r) = 0 \qquad r \to \infty$$

- Looking for the solution

$$\chi(r) = \frac{A(r,\theta,\varphi)}{r}$$

$$A'' + k^2 A = 0 \qquad A'' = \frac{d^2A}{dr^2}$$

Solution of the above equation

$$A_{\pm}(r,\theta,\varphi) = \frac{f(\theta,\varphi)}{(2\pi)^{3/2}} e^{\pm ikr} \qquad \left(\begin{matrix} kr \\ \text{no} \\ \vec{k}\cdot\vec{r} \end{matrix}\right)$$

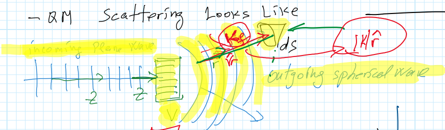
- with final result $\quad \chi_{\pm}(r) = \frac{f(\theta,\varphi)}{(2\pi)^{3/2}}\frac{e^{\pm kr}}{r}$

- Choose Diverging Spherical Wave

$\chi_+(r) = f(\theta,\varphi)\, e^{+ikr}$ ...

$$\chi_+(r) = \frac{f(\theta,\varphi)}{(2\pi)^{3/2}} \frac{e^{ikr}}{r} \Big|_{r\to\infty}$$

– QM Scattering Looks Like



– Calculated $\vec{J}_{out} = -\frac{i}{2m}(\chi_+^* \vec{\nabla}\chi_+ - \chi_+\vec{\nabla}\chi_+^*)$!

$$\vec{J}_{out} = \frac{|f(\theta,\varphi)|^2}{(2\pi)^3 r^2}\hat{v}_r$$

target 10^{-13} cm

10^{-4} cm

$$\chi_+(r) = \frac{f(\theta,\varphi)}{(2\pi)^{3/2}} \frac{e^{ikr}}{r}$$

$$\vec{\nabla}\chi_+ = \frac{\partial}{\partial x}\chi(r)\hat{x} + \frac{\partial}{\partial y}\chi(r)\hat{y} + \frac{\partial}{\partial z}\chi(r)\hat{z}$$

$$\left[\begin{array}{l} \frac{\partial}{\partial x}\chi(r) = \frac{\partial \chi(r)}{\partial r}\frac{\partial r}{\partial x} \\ \partial y \\ \partial z \end{array}\right]$$

– Calculated QM Diff. Cross Section

$$d\sigma^{QM} = \frac{\vec{J}_{out}\cdot d\vec{s}}{J_{inc}} = |f(\theta,\varphi)|^2 d\Omega$$

theory

$$\frac{d\sigma^{QM}}{d\Omega} = |f(\theta,\varphi)|^2 \quad ?$$

$$ds = d\Omega\, r^2$$

$$\frac{d\sigma^{exp}}{d\Omega}$$

⇒ $f(\theta,\varphi)$ – Scattering Amplitude

↳ It contains Information about $V(r)$

⇒ To Calculate $f(\theta,\varphi)$ – Concentrate on

$$(\nabla^2 + k^2)\chi(r) = U(r)\Psi(r) \quad (1)$$

and consider $r\to\infty$ limit

where $U(r) = 2mV(r)$

?

$$\Psi(r) = \phi_k(r) + \chi(r)$$

⇒ Green Function Method

Solving above Diff. Equation Using Green Function Method

– suppose we know

$$[\nabla^2 + k^2]G(r,r') = \delta^3(r'-r)$$

– Then we can write the solution of eq(1) as

$$(2)\quad \chi(r) = \int G(r,r')U(r')\Psi(r')d^3r' \;!!$$

⇒ Checking above statement, insert $\chi(r)$ into Eq(1)

$$(\nabla^2+k^2)\chi(r) = U(r)\Psi(r)$$

$$(\nabla^2+k^2)\int G(r,r')U(r')\Psi(r')d^3r' = \int \delta^3(r-r')U(r')\Psi(r')d^3r' =$$

$$\boxed{\int f(r')\delta(r-r')dr' = f(r)}$$

$$= U(r)\Psi(r)$$

$G(r,r')$ – is called Green Function

– not unique

→ we chose one that at $r\to\infty$ describes diverging spherical wave

$$(3)\quad G(r,r') = -\frac{e^{ik|r-r'|}}{4\pi|r-r'|}$$

HW !

⇒ From Eq(2) and (3) we have

$$(4)\quad \chi(r) = -\frac{1}{4\pi}\int \frac{e^{ik|r'-r|}}{|r'-r|}U(r')\Psi(r')d^3r'$$

confined by R $|r'| < |R|$

$U(r')\to 0$ $r' > R$

$r\to\infty$

$$\chi(r)\Big|_{r\to\infty} = \boxed{\frac{f(\theta,\varphi)}{(2\pi)^{3/2}}\frac{e^{ikr}}{r}}$$

scattering amplitude

(is defined $f(\theta,\varphi)$)

↳ consider this relation in the limit of $r\to\infty$

Remember that once $r > R$ $V(r) = 0$

↳ Therefore $U(r') = 0$ once $r' > R$

→ Therefore only region of $r' < R$ contributes in the integral of Eq(4)

↳ Thus $\boxed{r' < R}$ in Eq(4)

– Consider now Eq(4) for $r \gg R$ $r\to\infty$

$$|\vec{r}'-\vec{r}| = \sqrt{r^2 - 2\vec{r}\vec{r}' + r'^2} = r\sqrt{1 - \frac{2\vec{r}\vec{r}'}{r^2} + \cancel{\left(\frac{r'}{r}\right)^2}} = \sqrt{1 - \frac{2\vec{r}\vec{r}'}{r^2}}$$

$$\sqrt{1+x}\Big|_{x\ll 1} \approx \left(1 + \frac{x}{2}\right)$$

Since $\frac{r'}{r} \ll 1$

$$= |r'-r| = r\left(1 - \frac{\vec{r}\vec{r}'}{r^2}\right) = \boxed{r - \frac{\vec{r}\cdot\vec{r}'}{r}}$$

$$\left(\frac{R}{r}\right) \sim \Theta\left(\frac{R}{r}\right)$$

– insert $|r'-r|$ into Eq(4)

$$\chi(r)\Big|_{r\to\infty} = -\frac{1}{4\pi}\int \frac{e^{ik\left(r - \frac{\vec{r}\vec{r}'}{r}\right)}}{r - \cancel{\frac{\vec{r}\vec{r}'}{r}}}U(r')\Psi(r')d^3r' =$$

– analyse.

$$e^{ik\left(r - \frac{\vec{r}\vec{r}'}{r}\right)} = e^{ikr}\cdot e^{-\frac{ik\vec{r}\vec{r}'}{r}} =$$

$r - \frac{r'}{r} = r\left(1 - \frac{r'}{r^2}\right)$

denominator

$$\chi(r)\Big|_{r\to\infty} = -\frac{1}{4\pi}\int \frac{e^{ikr}}{r} e^{-ik\hat{r}\cdot\vec{r}'} U(r')\psi(r')d^3r' =$$

$$= -\frac{1}{4\pi}\frac{e^{ikr}}{r}\int e^{-i\frac{k\vec{r}\cdot\vec{r}'}{r}} U(r')\psi(r')d^3r' \quad \rightarrow \vec{k}_f\cdot\vec{r}'$$

— $k\frac{\vec{r}}{r}\vec{r}' = |k|\hat{r}\cdot\vec{r}'$ ($\hat{r}$; $k \to r$ direction) | introduce $\vec{k}_f = \vec{k}' = |k|\cdot\hat{r}$

$$\boxed{\chi(r)}\Big|_{r\to\infty} = -\frac{1}{4\pi}\frac{e^{ikr}}{r}\int e^{-i\vec{k}_f\vec{r}'} U(r')\psi(r')d^3r' \quad ⓐ$$

— Remember Definition of $\boxed{\chi(r)\Big|_{r\to\infty}} = \boxed{\frac{f(\theta,\varphi)}{(2\pi)^{3/2}}\frac{e^{ikr}}{r}}$ ⓑ

$\psi(r') = \phi_k(r) + \chi(r)$ — contained $f(\theta,\varphi)$

— Compare ⓐ and ⓑ $\phi^*_{k_f}(r)$

$$\Rightarrow \boxed{f(\theta,\varphi) = -\frac{(2\pi)^{3/2}}{4\pi}\int e^{-i\vec{k}_f\cdot\vec{r}'} U(r')\psi(r')d^3r'}$$

$\boxed{\vec{k}_f = |k|\hat{r}}$

$\boxed{\frac{d\sigma}{d\Omega} = |f(\theta,\varphi)|^2}$

— Wave function of free particle with momentum k_f is $\boxed{\phi_{k_f}(r) = \frac{1}{(2\pi)^{3/2}}e^{i\vec{k}_f\vec{r}}}$

$\boxed{\phi_k(r) = \frac{1}{(2\pi)^{3/2}}e^{ikr}}$

$$\boxed{f(\theta,\varphi) = -\frac{(2\pi)^3}{4\pi}\int \phi^*_{k_f}(r')U(r')\psi(r')d^3r = -2\pi^2\langle\phi_{k_f}|U|\psi\rangle}$$

$\phi_k(r) + \chi(r)$ contained f

⇒ Perturbation Theory: Born Series

$$f(\theta,\varphi) = -2\pi^2\int \phi^*_{k_f}(r)\,U(r)\,\boxed{\psi(r)}\,d^3r' \quad ⑤$$

$r' \equiv r$

first approx $\psi(r') = \phi_k(r')$

Remember $\psi(r) = \phi_{k_i}(r) + \chi(r)$ ⑥

where $\chi(r)\Big|_{r\to\infty} = \frac{f(\theta,\varphi)}{(2\pi)^{3/2}}\frac{e^{ikr}}{r}$

$\chi(r) \ll \phi_{k_i}(r)$

— Problem of calculating $f(\theta,\varphi)$ in Eq (5)

to calculate $f(\theta,\varphi)$ one needs to know $f(\theta,\varphi)$

— Perturbation: Lets assume that $\boxed{V(r)}$ is a weak field

Such that $f(\theta,\varphi) \sim U(r) \sim 2mV(r)$ – is small

— Such that $\boxed{\chi(r)} \ll \boxed{\phi_{k_i}(r)}$ $\chi(r) \sim f(\theta,\varphi)$

— Then in the $\boxed{\text{first}}$ approximation in Eq (5)

we take $\boxed{\psi(r') \simeq \phi_{k_i}(r)}$ and obtain

$$\boxed{f^{(1)}(\theta,\varphi) = -2\pi^2\int \phi^*_{k_f}(r')\,U(r')\,\phi_{k_i}(r')\,d^3r'}$$

— Once we calculated $f^{(1)}(\theta,\varphi)$

we construct

$$\psi(r) = \phi_k(r) + \chi^{(1)}(r)$$

where $\chi^{(1)} = \frac{f^{(1)}(\theta,\varphi)}{(2\pi)^{3/2}}\frac{e^{ikr}}{r}$

and calculate

and calculate

$$f^{(2)} = -2\pi^2 \int \phi^*_{k_f}(r') U(r') \chi^{(1)}(r') d^3r'$$

— Then construct $\chi^{(2)} = \frac{f^{(2)}}{(2\pi)^{3/2}} \frac{e^{ikr}}{r}$

calculate

$$f^{(3)} = -2\pi^2 \int \phi^*_{k_f}(r') U(r') \chi^{(2)}(r') d^3r'$$

— Eventually the complete Scattering amplitude is

$$f(\theta,\varphi) = f^{(1)}(\theta,\varphi) + f^{(2)}(\theta,\varphi) + f^{(3)}(\theta,\varphi) + \dots$$

Feynman Diagrams

$\Rightarrow$ Born Approximation

in many cases $f^{(3)} << f^{(2)} << f^{(1)}$ and it is enough to calculate the first term only

$\alpha = \frac{1}{137}$

$\alpha^2 f^{(1)}$

$e^2 \sim \hbar c$

$\alpha^4 f^{(2)}$

— The first term is called $\boxed{\text{Born Approximation}}$

$$f^B \equiv f^{(1)} = -2\pi^2 \int \phi^*_{k_f}(r') U(r') \phi_{k_i}(r') d^3r'$$

$$\boxed{\phi_k = \frac{1}{(2\pi)^{3/2}} e^{i\vec{k}\vec{r}}}$$

$$f^{(1)}(\theta,\varphi) = -\frac{2\pi^2}{(2\pi)^3} \int e^{-i\vec{k}_f \cdot \vec{r}'} U(r') e^{i\vec{k}_i \cdot \vec{r}'} d^3r' =$$

$r' = r$

$$= -\frac{1}{4\pi} \int e^{+i(\vec{k}_i - \vec{k}_f)\vec{r}'} U(r') d^3r' = //$$

remember $\boxed{\vec{q} = \vec{k}_i - \vec{k}_f}$

$$// = f^{(1)}(\theta,\varphi) = -\frac{1}{4\pi} \int e^{+i\vec{q}\vec{r}'} U(r') d^3r' = -\frac{m}{2\pi} \int e^{+i\vec{q}\vec{r}'} V(r') d^3r'$$

$\boxed{U(r) = 2mV(r)}$

— introduce $\boxed{V(q) = \int e^{+i\vec{q}\cdot\vec{r}} V(r) d^3r}$ Fourier Transform

$$f^{(1)}(\theta,\varphi) = -\frac{m}{2\pi} V(q)$$?

Born Approximation

$$— \frac{d\sigma^{el}}{d\Omega} = |f^{(1)}(\theta,\varphi)|^2$$

Examples

$$\frac{d\sigma}{d\Omega} = \frac{d\sigma_R}{d\Omega}\left(1 + \pi\alpha \sin\left(\frac{\theta}{2}\right)\right)$$