L20

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- Identical Particles
- Many Particle Systems — solids, stars, gases, ...
- Principle of indistinguishable Particles

Permutation operator $\longrightarrow \hat{P}_{12} |\psi_{12}\rangle = |\psi_{21}\rangle$ | $\hat{P}_{ij} |\psi_{1,2,3\cdots i,j\cdots n}\rangle = |\psi_{123\cdots j\, i\cdots n}\rangle$

- if particles are indistinguishable then their hamiltonian $\hat{H}$ is commuting with the permutation operator $\hat{P}_{ij}$ - which permutes particles $i \rightleftarrows j$

$\hat{H}_{12}$

- $[\hat{H}_{12}, \hat{P}_{ij}] = 0$, Therefor they should have same state vectors
- $\hat{P}_{ij} \cdot \psi(r_1 \cdots r_i \cdots r_j \cdots r_N) = \psi(r_1 \cdots r_j \cdots r_i \cdots r_N)$ $\quad \hat{A}|\psi\rangle = a|\psi\rangle$

$\boxed{\hat{P}_{ij} \cdot \hat{P}_{ij}} \psi(r_1 \cdots r_i \cdots r_j \cdots r_N) = 1 \psi(r_1 \cdots r_i \cdots r_j \cdots r_N)$

$\hat{P}_{ij}^2 = I$ — eigenvalue of $\hat{P}_{ij}^2 \Rightarrow 1$, $\hat{P}_{ij}$ — eigenvalue 1, -1

- $\boxed{\hat{P}_{ij} \psi_\pm (r_1, r_2 \cdots r_i \cdots r_j \cdots r_N) = \pm \psi_\pm (r_1, r_2 \cdots r_i \cdots r_j \cdots r_N)}$
- Fermions are described by ψ_-,
  Bosons are described by ψ_+
- Example for two electron Wave Function

$$\boxed{\psi_-(r_1, r_2) = \frac{\psi_1(r_1)\psi_2(r_2) - \psi_1(r_2)\psi_2(r_1)}{\sqrt{2}}}$$

$$P_{12} \psi_-(r_1, r_2) = -\frac{\psi_1(r_2)\psi_2(r_1) - \psi_1(r_1)\psi_2(r_2)}{\sqrt{2}} =$$

$$= -\left(\frac{\psi_1(r_1)\psi_2(r_2) - \psi_1(r_2)\psi_2(r_1)}{\sqrt{2}}\right) = -\psi_-(r_1, r_2)$$

- Example for two boson particles

$$\boxed{\psi_+(r_1, r_2) = \frac{\psi_1(r_1)\psi_2(r_2) + \psi_1(r_2)\psi_2(r_1)}{\sqrt{2}}}$$

$$P_{12} \psi_+(r_1, r_2) = \frac{\psi_1(r_2)\psi_2(r_1) + \psi_1(r_1)\psi_2(r_2)}{\sqrt{2}} = +\psi_+(r_1, r_2)$$

- Example for 3 System $\quad P_{ij} \psi_-(r_1 r_2 r_3) = -\psi_-(r_1 r_2 r_3)$

$$\psi_-(r_1, r_2, r_3) = \psi_1(r_1)\psi_2(r_2)\psi_3(r_3) + \psi_3(r_1)\psi_1(r_2)\psi_2(r_3)$$
$$+ \psi_2(r_1)\psi_3(r_2)\psi_1(r_3) - \psi_1(r_1)\psi(r_2)\psi_2(r_3)$$
$$- \psi_3(r_1)\psi_2(r_2)\psi_1(r_3) - \psi(r_1)\psi_1(r_2)\psi_3(r_3)$$

$$= \begin{vmatrix} \psi_1(r_1) & \psi_1(r_2) & \psi_1(r_3) \\ \psi_2(r_1) & \psi_2(r_2) & \psi_3(r) \\ \psi_3(r_1) & \psi_3(r_2) & \psi_3(r_3) \end{vmatrix} \sim 3! \text{ terms}$$

$$\psi_+(r_1 r_2 r_3) = \psi_1(r_1)\psi_2(r_2)\psi_3(r_3) + \psi_3(r_1)\psi_1(r_2)\psi_2(r_3)$$
$$\psi_2(r_1)\ \psi_3(r_2)\psi_1(r_3) + \psi_1(r_1)\psi_3(r_2)\psi_2(r_3) \sim 3! \text{ terms}$$
$$\psi_3(r_1)\psi_2(r_2)\psi_1(r_3) + \psi_2(r_1)\psi_1(r_2)\psi_3(r_3)$$

Carbon → 12!

- For N particle system

$$\psi_-(r_1 \cdots r_N) = \begin{vmatrix} \psi_1(r_1) \cdots \psi_1(r_i) \cdots \psi_1(r_N) \\ \psi_j(r_1) \cdots \psi_j(r_i) \cdots \psi_j(r_N) \\ \psi_N(r_1) \cdots \psi_N(r_i) \cdots \psi_N(r_N) \end{vmatrix} \sim N! \text{ terms}$$

Fermions

Slater Determinant

10^{26-34}

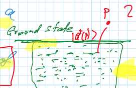
$$\psi_+(r_1 \cdots r_N) = \{\psi_1(r_1) \cdots \psi_1(r_i) \cdots \psi_1(r_N)\}_{\text{sym permutations}} \sim N! \text{ terms}$$

- Above approach is intractable for $N \sim 10^{26}$! QM

- New Approach — Second Quantization

1st quantization
quantize – Energy
Momentum

① Ground State of the Universe is a Reservoir of matter which is Quantized by discrete number of Particles



2nd Quantization
quantize Vacuum

- quantum state vector of the ground state

$|0\rangle$ or $|\ \rangle$ $\quad$ Normalization $\langle | \rangle = 1$

$\langle | \rangle = 1$

② Define Creation Operators such that

definition
$$\begin{cases} \boxed{a^+(p)|\ \rangle} = |p\rangle_B \text{ — Free Boson with Momentum } \vec{p} \\ b^+(p)|\ \rangle = |p\rangle_F \text{ — Free Fermion with Momentum } \vec{p} \end{cases}$$

③ Define these operators Such a way that

Annihilation operators $\Rightarrow$ $\begin{cases} a(p)|0\rangle = 0 \\ b(p)|0\rangle = 0 \end{cases}$ 1st. Property

(4) Prove that for Free Particles

Eq. (1)
$$\left.\begin{aligned} &[a(p), a(p')] = 0 \\ &[a^+(p)\, a^+(p')] = 0 \\ &[a(p), a^+(p')] = \delta_3(p'-p) \end{aligned}\right\} \text{For Bosons}$$

anticommutator (2)
$$\left.\begin{aligned} &\{b(p), b(p')\} = b(p)\, b(p') + b(p')\, b(p) = 0 \\ &\{b^+(p), b^+(p')\} = 0 \\ &\{b(p), b^+(p')\} = \delta_3(p'-p) \end{aligned}\right\} \text{For Fermions}$$

(5) For Bound states of particles state levels i, j descrete numbers

$$\left.\begin{aligned} &[a_i\ a_j] = 0 \\ &[a_i^+\ a_j^+] = 0 \\ &[a_i\ a_j^+] = \delta_{ij} \end{aligned}\right\} \text{Bosons} \qquad \left.\begin{aligned} &\{b_i\, b_j\} = 0 \\ &\{b_i^*\, b_j^+\} = 0 \\ &\{b_i\ b_j^+\} = \delta_{ij} \end{aligned}\right\} \text{Fermions}$$

$\Rightarrow$ Prove of Eq (1)

$|0\rangle \equiv |\rangle$ – We know previously $\boxed{\langle p'|p\rangle = \delta^3(\vec{p}' - \vec{p})}$

$a^+(p)|0\rangle = |p\rangle$ using the definition of $\boxed{a^+(p)|\rangle = |p\rangle}$

$\langle p' = \langle 0|a(p')$

$\langle p'|p\rangle = \boxed{\langle|a(p')\, a^+(p)|\rangle = \delta^3(p-p')}$ $\langle|\rangle = 1$

$\langle p'|p\rangle$

Possible solutions of this equation

(a) $\boxed{a(p')\, a^+(p) = \delta^3(p-p')}$

or

(b) $\boxed{a(p')\, a^+(p) = \delta^3(p-p') + A\, a^+(p)\, a(p')}$

$\langle|\delta^3(p-p') + A a^+(p) a(p')|0\rangle = \langle|\delta^3(p-p')|\rangle + A\langle|a^+(p)\cancel{a(p)|\rangle} =$

$= \langle|\rangle \delta^3(p-p') = \delta^3(p-p')$

– to estimate A consider two particle state

$$\boxed{|p_1 p_2\rangle = \frac{|_1 p_1\rangle|_2 p_2\rangle + |_1 p_2\rangle |_2 p_1\rangle}{\sqrt{2}}} \quad \text{this we know}$$

HW

$$\boxed{\langle p_1' p_2' | p_1 p_2\rangle = \delta(p_1 - p_1')\,\delta(p_2 - p_2') + \delta(p_1 - p_2')\,\delta(p_2 - p_1')} \quad (3)$$

$\langle p|p'\rangle = \delta^3(p-p')$

– In second quantization approach

$$|p_1 p_2\rangle = a^+(p_1)\, a^+(p_2)|\rangle$$

(1) therefore

$$\langle p_1' p_2' | p_1 p_2\rangle = \langle|a(p_2')\, a(p_1')\, a^+(p_1)\, a^+(p_2)|0\rangle$$

if (a) is correct

$$\langle p_1' p_2' | p_1 p_2\rangle = \langle|a(p_2')\,\delta^3(p_1' - p_1)\, a^+(p_2)|\rangle = \boxed{\delta^3(p_2' - p_1')\,\delta^3(p_2 - p_1)}$$

(which disagrees with Eq (3))

for (b)

$$\langle p_1' p_2' | p_1 p_2\rangle = \langle|a(p_2')\left(\delta^3(p_1' - p_1) + A\, a^+(p_1)\, a(p_1')\right) a^+(p_2)|\rangle =$$

$$\langle|a(p_2')\,\delta^3(p_1' - p_1)\, a^+(p_2)|\rangle + A\langle|a(p_2')\, a^+(p_1)\, a(p_1')\, a^+(p_2)|\rangle$$

(1) $\boxed{a(p)|\rangle = 0}$ use

$$= \delta^3(p_1' - p_1)\,\delta^3(p_2' - p_2) + A\langle|(\delta^3(p_2' - p_1) + A a^+(p_1)\, a(p_2'))\, a(p_1')\, a^+(p_2)|\rangle$$

$$= \delta^3(p_1' - p_1)\,\delta^3(p_2' - p_2) + A\langle|\delta^3(p_2' - p_1)\, a(p_1')\, a^+(p_2)|\rangle + A^2\langle|a^+(p_1)\, a(p_2')\, a(p_1')\, a^+(p_2)|\rangle$$

$$= \delta^3(p_1' - p_1)\,\delta^3(p_2' - p_2) + A\langle|\delta^3(p_2' - p_1)\{\delta^3(p_1' - p_2) + A a^+(p_2)\, a(p_1')\}|\rangle =$$

$$= \delta^3(p_1' - p_1)\,\delta^3(p_2' - p_2) + A\,\delta^3(p_2' - p_1)\,\delta^3(p_1' - p_2)$$

Comparing this with Eq (3)

For (b) one obtains $\boxed{A = 1}$

$$\Rightarrow a(p')\, a(p)^+ = \delta^3(p' - p) + a(p)^+\, a(p')$$

or

$$\boxed{[a(p')\, a(p)^+] = \delta^3(p' - p)}$$

– Prove that $[a^+(p')\, a^+(p)] = 0$ Bosons

$a(p)|\rangle = 0$

$\langle|a^+(p) = 0$

$$(a^+(p')\, a^+(p) - a^+(p)\, a^+(p'))|\rangle =$$

$$= |p' p\rangle - |p p'\rangle = \frac{|p'\rangle|p\rangle + |p\rangle|p'\rangle}{\sqrt{2}} - \frac{|p\rangle|p'\rangle + |p'\rangle|p\rangle}{\sqrt{2}} = 0$$

from $[a^+(p')\, a^+(p)] = 0 \Rightarrow [a(p')\, a(p)] = 0$

$[a(p), a(p')] = 0$

– Summarizing

For Bosons

$$\begin{cases} [a(p)\, a^+(p')] = \delta^3(p' - p) \\ [a(p)\ a(p')] = 0 \\ [a^+(p),\ a^+(p')] = 0 \end{cases}$$

– Other operators

– Number operator $\hat{N} = \int a^\dagger(p) a(p)\, d^3p$

gives the total number of the particles in the state

$\hat{N}|p_1p_2p_3\rangle = 3|p_1p_2p_3\rangle$

$\hat{N}|p_1p_2\rangle = 2|p_1p_2\rangle$

– Example

$$\hat{N}|p_1p_2\rangle = \int a^\dagger(p)a(p)\,|p_1p_2\rangle\, d^3p = \qquad |p_1p_2\rangle = a^\dagger(p_1)a^\dagger(p_2)|\,\rangle$$

$a(p)|\,\rangle = 0$

$$= \int a^\dagger(p)\,a(p)\,a^\dagger(p_1)\,a^\dagger(p_2)|\,\rangle\, d^3p = \qquad \boxed{[a(p)a^\dagger(p_1)] = \delta^3(p-p')}$$

$a(p)a^\dagger(p_1) = \delta(\;) + a^\dagger(p_1)a(p)$

$$= \int a^\dagger(p)\left[\delta^3(p-p_1) + a^\dagger(p_1)a(p)\right]a^\dagger(p_2)|\,\rangle\, d^3p =$$

$$= \int a^\dagger(p)\,a^\dagger(p_2)\,\delta^3(p-p_1)|\,\rangle\, d^3p + \int a^\dagger(p)\,a^\dagger(p_1)\,a(p)\,a^\dagger(p_2)|\,\rangle\, d^3p$$

$$= a^\dagger(p_1)\,a^\dagger(p_2)|\,\rangle + \int a^\dagger(p)\,a^\dagger(p_1)\left[\delta^3(p-p_2) + a^\dagger(p_2)a(p)\right]|\,\rangle\, d^3p$$

$$= |p_1p_2\rangle + \int a^\dagger(p)\,a^\dagger(p_1)\,\delta^3(p-p_2)|\,\rangle\, d^3p + \int a^\dagger(p)a^\dagger(p_1)a^\dagger(p_2)\,a(p)|\,\rangle\, d^3p \to 0$$

$$= |p_1p_2\rangle + \underbrace{a^\dagger(p_2)\,a^\dagger(p_1)|\,\rangle}_{a^\dagger(p_1)\,a^\dagger(p_2)} = |p_1p_2\rangle + |p_1p_2\rangle = 2|p_1p_2\rangle$$

Thus $\hat{N}|p_1p_2\rangle = 2|p_1p_2\rangle$

– Using Number operator we define

$$\hat{H} = \int \omega(p)\, a^\dagger(p)a(p)\, d^3p$$

$\hbar = 1$

$\boxed{\omega(p) = \frac{p^2}{2m}}$ – free particles

$\hat{H}|p\rangle = \frac{p^2}{2m}|p\rangle \qquad \hat{H}|p\rangle = \varepsilon|p\rangle$

$\hat{H}|p_1p_2\rangle = \left(\frac{p_1^2}{2m} + \frac{p_2^2}{2m}\right)|p_1p_2\rangle$ HW $\quad \hat{H}|p_1p_2\rangle = (\varepsilon_1+\varepsilon_2)|p_1p_2\rangle$

⇒ Considering Fermions

⟹ Very similar as above

$\boxed{\hat{P}_{12}|p_{12}\rangle = -|p_{12}\rangle}$

$$|p_1p_2\rangle = \frac{|p_1\rangle|p_2\rangle - |p_2\rangle|p_1\rangle}{\sqrt{2}} \qquad \boxed{p_1 = p_2}$$

$$\langle p_1'p_2'|p_1p_2\rangle = \delta^3(p_1-p_1')\,\delta^3(p_2-p_2') - \delta^3(p_1-p_2')\,\delta^3(p_2-p_1') \qquad \langle p_1|p_2\rangle = \delta^3(p_1-p_2)$$

Obtain

$$\{b(p)\,b^\dagger(p')\} = \delta^3(p'-p)$$

$$\{b(p)\;b(p')\} = 0$$

$$\{b^\dagger(p),\; b^\dagger(p')\} = 0$$

Fermions

– Number Operator

$$\boxed{\hat{N} = \int b^\dagger(p)\,b(p)\, d^3p}\;!$$

– Hamiltonian Operator

$$\hat{H} = \int \omega(p)\, b^\dagger(p)\, b(p)\, d^3p$$

$|p_1p_2\rangle = b^\dagger(p_1)\,b^\dagger(p_2)|\,\rangle$

$\{b(p)\; b^\dagger(p_1)\} = \delta^3(p-p_1)$

$b(p)b^\dagger(p_1) + b^\dagger(p_1)\,b(p) = \delta^3(p-p_1)$

$b(p)b^\dagger(p_1) = \delta^3(p-p_1) - b^\dagger(p_1)b(p)$

$$\hat{N}|p_1p_2\rangle = \int b^\dagger(p)\,b(p)\,b^\dagger(p_1)\,b^\dagger(p_2)|\,\rangle\, d^3p =$$

$$= -\int b^\dagger(p)\,b^\dagger(p_1)\,b(p)\,b^\dagger(p_2)|\,\rangle\, d^3p =$$

$$+\int b^\dagger(p)\,\delta^3(p-p_1)\,b^\dagger(p_2)|\,\rangle\, d^3p =$$

$$= b^\dagger(p_1)\,b^\dagger(p_2)|\,\rangle - \int b^\dagger(p)\,b^\dagger(p_1)\,\delta^3(p-p_2)|\,\rangle\, d^3p$$

$$= \underbrace{|p_1p_2\rangle}_{1\rightleftharpoons 2} - \underbrace{|p_2p_1\rangle}_{1\rightleftharpoons 2} = 2|p_1p_2\rangle$$

$-|p_2p_1\rangle = |p_1p_2\rangle$

⇒ Bound Particles ⇒ spectrum is discrete

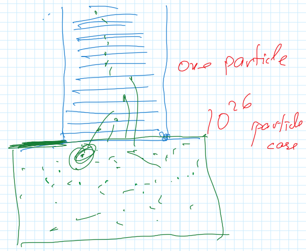
Bosons $\boxed{a_i^+|\rangle = |n_i\rangle}$

$\boxed{a_i|\rangle = 0}$

$$\begin{cases} [a_i, a_j^+] = \delta_{ij} \\ [a_i, a_j] = 0 \\ [a_i^+, a_j^+] = 0 \end{cases}$$

Fermions $b_i^+|\rangle = |n_i\rangle$

$$\begin{cases} \{b_i, b_j^+\} = \delta_{ij} \\ \{b_i, b_j\} = \{b_i^+, b_j^+\} = 0 \end{cases}$$



$\Rightarrow$ Number operator

Bosons $\hat{N} = \sum_i a_i^+ a_i$

Fermions $\hat{N}_F = \sum_i b_i^+ b_i$

Hamiltonian

Bosons $H_B = \sum_i \varepsilon_i a_i^+ a_i$

Fermions $H_F = \sum_i \varepsilon_i b_i^+ b_i$