L26

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Bose Einstein Condensation

- Integer spin particles with large mass
- Particles make the Bose-Einstein gas
- Example of such particle is ^4He atoms
- Quantum Effects expected at very low T $\longrightarrow$ Bose-Einstein Condensation
- BEC - related to superfluidity - Liquid Helium
- No Exclusion Principle
- Consider Ideal Boson gas consisting of N-particles in the volume V at T

$$f(\varepsilon) = \frac{N(\varepsilon)}{g(\varepsilon)} = \frac{1}{e^{\frac{(\varepsilon-\mu)}{kT}} - 1} \qquad (1)$$

1) μ as it depends on T

- Adopt convention of Ground State Energy to be 0
- At $T=0$ all Bosons N are in the Ground state
- Setting $\varepsilon = 0$ ($\varepsilon \to 0$) $f(0)$ makes sense if μ is negative
- at $T=0$ $\mu = 0$ and $\mu < 0$ $T > 0$
- At high temperatures - in the Classical Limit $\longrightarrow$ Maxwell Boltzmann distribution

$$f(\varepsilon) = e^{\frac{-(\varepsilon-\mu)}{kT}} \quad \text{- check the}$$

- $$Z = \sum_j g_j e^{-\varepsilon_j/kT} = \int_0^\infty g(\varepsilon) e^{-\varepsilon/kT} \, dn$$

$$e^{-\frac{\varepsilon}{kT}} \cdot e^{\frac{\mu}{kT}} = f(\varepsilon) = \frac{N(\varepsilon)}{g(\varepsilon)}$$

$$g(\varepsilon) e^{-\frac{\varepsilon}{kT}} \cdot e^{\frac{\mu}{kT}} = N(\varepsilon)$$

$$\int g(\varepsilon) e^{-\frac{\varepsilon}{kT}} \, dn \cdot e^{\frac{\mu}{kT}} = \int N(\varepsilon) \, d\varepsilon = N$$

$$e^{\mu/kT} = \frac{N}{Z}$$

$$\frac{\mu}{kT} = \ln\frac{N}{Z}, \quad \boxed{\mu = -kT\ln\frac{Z}{N}}$$

– phase space of non-relativistic Particle

$$dn = V\frac{d^3p}{(2\pi\hbar)^3} = \frac{Vp^2\,dp\,d\Omega}{(2\pi\hbar)^3} = \qquad P = \sqrt{2m\varepsilon}$$

$$= \frac{V\,2m\varepsilon\sqrt{2m}\cdot\frac{1}{2}\frac{d\varepsilon}{\sqrt{\varepsilon}}\,d\Omega}{(2\pi\hbar)^3} = \frac{4\pi V\sqrt{2}\cdot m^{3/2}\varepsilon^{1/2}d\varepsilon}{h^3} =$$

$$g(\varepsilon)dn = \gamma_s\cdot V\cdot\frac{4\sqrt{2}\pi}{h^3}m^{3/2}\varepsilon^{1/2}d\varepsilon$$

– Calculate Z $\quad \gamma_s = 1$

$$Z = V\cdot\frac{4\sqrt{2}\pi}{h^3}m^{3/2}\int\varepsilon^{\frac{1}{2}}e^{-\frac{\varepsilon}{kT}}\,d\varepsilon$$

HW

$$Z = \left(\frac{2\pi mkT}{h^2}\right)^{3/2}$$

$$\rightarrow \frac{\mu}{kT} = -\ln\left[\left(\frac{2\pi mkT}{h^2}\right)^{3/2}\frac{V}{N}\right] \quad 2$$

– One kilomole of Boson gas of ^4He at $T = 273$

$$m_H = \frac{6.65\times10^{-27}\cdot(1.38\times10^{-23})\cdot 273}{(6.63\times10^{-34})^2}\left(\frac{22.4}{6.02\times10^{26}}\right)$$

$$\mu = -0.23\,eV$$

– Comparing with average energy of monoatomic gas $\varepsilon = (\frac{3}{2})kT = 0.035\,eV \quad 273K$

– So $\frac{(\varepsilon-\mu)}{kT} = 1.5 + 12.4 = 13.9$

– Substituting in $f(\varepsilon) = e^{-\frac{(\varepsilon-\mu)}{kT}} = \frac{9.2\times10^{-7}}{\text{Dilute Gas}}$

Dilute Gas Assumption is Valid

— from (2)

$$e^{-\frac{\mu}{kT}} = \left(\frac{2\pi m k T}{h^2}\right)^{3/2} \frac{V}{N}$$

$\Rightarrow$ Increases with T

$\Rightarrow$ Decreases with $\rho_N = \frac{N}{V}$

— Distribution of N with Temperature

μ - function of N/V

$\Rightarrow$ Continuum approximation

$$\int_0^\infty N(\varepsilon)\, d\varepsilon = \int_0^\infty f(\varepsilon)\, g(\varepsilon)\, d\varepsilon = N$$

$$g(\varepsilon)\, d\varepsilon = \frac{4\sqrt{2}\,\pi V}{h^3} m^{3/2} \varepsilon^{1/2} d\varepsilon$$

$$N = 2\pi V \left(\frac{2m}{h^2}\right)^{3/2} \int \frac{\varepsilon^{1/2} d\varepsilon}{e^{\frac{(\varepsilon-\mu)}{kT}} - 1} \longrightarrow \text{Numerically} \quad (3)$$

$\Rightarrow$ problem that this does not account the $\varepsilon = 0$ contributio

this is N_{ex}

— Introduce $N = N_0 + N_{ex}$

— at $T \to 0$ $N_0 \approx N$

we can Put $\varepsilon = 0$ in Eq (1)

$$f(\varepsilon)^{T\to 0} = \frac{1}{e^{-\frac{\mu}{kT}} - 1} \approx N$$

since all particles are at $\varepsilon = 0$

$$- \frac{\mu}{kT} = \ln\left(1 + \frac{1}{N}\right) \approx \frac{1}{N}$$

— so $-\frac{\mu}{kT} \sim \frac{1}{N}$ — $\longrightarrow 0$ small

Thus in (3) we put $\frac{\mu}{kT}=0$ $T \to 0$ $x=\frac{\varepsilon}{kT}$

$$N_{ex} = V\frac{2}{\sqrt{\pi}}\left(\frac{2\pi m kT}{h^2}\right)^{3/2} \underbrace{\int \frac{x^{\frac{1}{2}}\,dx}{e^x-1}}_{2.612\,\frac{\sqrt{\pi}}{2}}$$

$$N_{ex} = 2.612 \cdot V\left(\frac{2\pi mkT}{h^2}\right)^{3/2} \quad (4)$$

— Bose Temperature T_B $N_{ex} = N$

$$N = 2.612\, V\left(\frac{2\pi mkT_B}{h^2}\right)^{3/2} \quad (5)$$

$$T_B = \frac{h^2}{2\pi mk}\left(\frac{N}{2.612\,V}\right)^{2/3}$$

— $T > T_B$ all Bosons are excited

— $T < T_B$ Bosons $\longrightarrow \varepsilon=0$ state

until all $N = N_0$

$$\frac{N_0}{N} = 1 - \frac{N_{ex}}{N} \qquad \text{from 4 and 5}$$

$$\frac{N_0}{N} = 1 - \left(\frac{T}{T_B}\right)^{3/2}$$

Kilomole

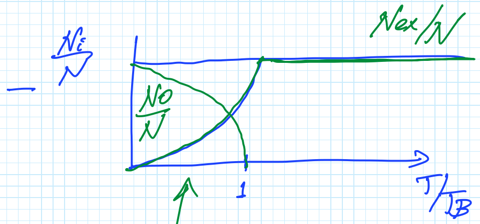
— Boson Gas ^4He 6.02×10^{23} ^4He

Volume $22.4 \times 10^{-3}\, m^3$

$m_{He} = 6.65 \times 10^{-27}\, kg$

$T_B = 0.036\,K$ HW

— ^4He Liquidified $T_L = 4.21\,K$ 1 atm



Bose Einstein Condensation

– Found in 1995

⟹ Properties of Boson Gas

– In the ground state do not contribute to the internal energy and nor to Heat Capacity

– $E = N_0 \varepsilon_0 + N_{ex} \varepsilon_{ex}$

for $T < T_B$, N_0 large but $\varepsilon_0 = 0$

for $T > T_B$, $N_0 = 0$

– For $T > T_B$ all N-s are in excitated state

Expect $U \to \frac{3}{2} N k T$

– For $T < T_B$ $\quad N_{ex} = N\left(\frac{T}{T_B}\right)^{3/2}$

– Assume each boson has thermal energy of the order of kT

$$U \simeq NkT\left(\frac{T}{T_B}\right)^{3/2} \quad (T < T_B)$$

$$U \sim T^{5/2}$$

— Nearly exact result obtained

$$U = \int \varepsilon N(\varepsilon) d\varepsilon$$

using $f(\varepsilon) = \frac{N(\varepsilon)}{g(\varepsilon)} = \frac{1}{e^{\frac{(\varepsilon-\mu)}{kT}} - 1}$ and

$$g(\varepsilon)\, d\varepsilon = \overset{\text{spin}}{\gamma_s} \frac{4\sqrt{2}\pi V}{h^3} m^{3/2} \varepsilon^{\frac{1}{2}} d\varepsilon$$

$$U = 2\pi V \left(\frac{2m}{h^2}\right)^{3/2} \int_0^\infty \frac{\varepsilon^{3/2} d\varepsilon}{e^{\frac{(\varepsilon-\mu)}{kT}} - 1}$$

— If we choose $\varepsilon_0 \simeq 0$ then $\mu \simeq 0$ for $T < T_B$

— Setting $\mu = 0$, substitution $x = \frac{\varepsilon}{kT}$

$$U = \frac{2}{\sqrt{\pi}} kT \left(\frac{2\pi m k T}{h^2}\right)^{3/2} V \underbrace{\int_0 \frac{x^{3/2} dx}{e^x - 1}}$$

$\Gamma\left(\frac{5}{2}\right) = \frac{3\pi^{1/2}}{4}$

Riemann Zeta

$\zeta\left(\frac{5}{2}\right) = 1.34$

— Using $T_B = \frac{h^2}{2\pi m k}\left(\frac{N}{2.61 V}\right)^{2/3}$

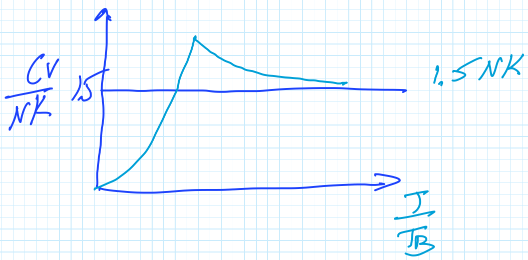
$$U = 0.77\, NkT \left(\frac{T}{T_B}\right)^{3/2} \quad (T < T_B)$$

— Heat Capacity:

$$C_v = \frac{dU}{\ldots} \quad 1.93\, Nk \left(\frac{T}{\ldots}\right)^{3/2} \quad (T < T_B)$$

$C_V = \frac{dU}{dT} = 1.92NK\left(\frac{T}{T_B}\right)$

$C_V \sim T^{3/2}$



— Entropy $T < T_B$

$$S = \int_0^T \frac{C_V\,dT'}{T'} = 1.28NK\left(\frac{T}{T_B}\right)^{3/2} \quad T < T_B$$

$S \to 0 \quad T \to 0$

— F — Helmholtz Function

$$F = U - TS$$

$$F = -0.51\,NKT\left(\frac{T}{T_B}\right)^{3/2} \quad (T < T_B)$$

— Pressure P

$$P = -\left(\frac{\partial F}{\partial V}\right)_{T,N}$$

$$T_B = \frac{h^2}{2\pi mK}\left(\frac{N}{2.612\,V}\right)^{2/3}$$

$$F = -1.33KT\left(\frac{2\pi mKT}{h^2}\right)^{3/2} \cdot V$$

$$P = 1.33KT\left(\frac{2\pi mKT}{h^2}\right)^{3/2} \quad (T < T_B)$$

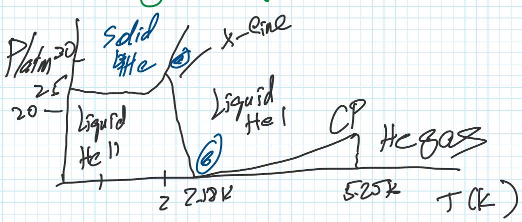
So $P \propto T^{5/2}$

independent of the volume

- This is because at $T = 0$ $\varepsilon = 0$ have no momentum and therefore make no contribution to the pressure.

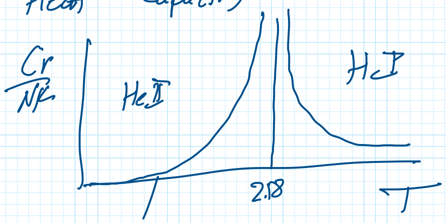
⇒ Application to Liquid Helium

Phase Diagram of ^4He



- Critical Point of 5.25K
- Unique Behavior at $T = 2.18K$
- Compressed isothermally $T > 2.18K$ it condenses to a liquid phase I
- Compressed at $T < 2.18K$ Liquid phase He II
- Helium II is superfluid
- λ-line I and II coexist
- He has 2 triple point a and b

— Heat Capacity



viscosity = 0 ⟹ superfluid

He II ideal Boson Gas

λ — Bose T_B

He II → No atoms

— Kilomole $V = 27 \times 10^{-3} m^3$

$\frac{N}{V} = \frac{6.02 \times 10^{26}}{27 \times 10^{-3}} = 2.2 \times 10^{28} m^{-3}$

^4He mass $= 6.63 \times 10^{-27} kg$

$T_B = 3.1K$

→ compared to 2.17 λ point