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Path Integration

⇒ Using Lagrangian Formalism to Formulate Quantum Mechanics

10.1.1. The Propagator as an Integral over Paths

- 1d particle is in the eigenstate of a position $|q_a\rangle$ at t_a

- The amplitude for it to be at $|q_b\rangle$ at a later time t_b is the propagator

  a) $U(t_b,t_a)|q_a\rangle = |\psi(t_b)\rangle$

  └ time evolution operator $U(t) = e^{-\frac{i\hat{H}}{\hbar}\cdot t}$

  b) for $|\psi(t_b)\rangle$ to be at $|q_b\rangle$ the amplitude is $\langle q_b|q_a\rangle$

  this is called a Propagator $K(q_b,t_b;q_a,t_a)$

  $$K(q_b,t_b;q_a,t_a) = \langle q_b|U(t_b,t_a)|q_a\rangle$$

  c) Should make it causal $\theta(t_b-t_a)$

  d) Avoid infinite time in calculation to put a factor $e^{-\mu}$ and take $\mu \to 0$ at the end of calculations

  b) $K(q_b,t_b;q_a,t_a) = \theta(t_b-t_a)\langle q_b|e^{-i(H(P,\hat{q})-i\eta)(t_b-t_a)}|q_a\rangle$

- Assume H - does not have Explicit t-dependence (stationary states)

  ① (assume $t_b > t_a$ - drop $\theta(\ )$ - as implicit)

  ② (remember $i\eta$ - to be restored at the end

  ③ Subdivide time into $N+1$ equal segments

  $t_0 = t_a$, $t_{N+1} = t_0 + (N+1)\varepsilon = t_b$ $\quad t_n = t_0 + n\varepsilon$

  ④ $U(t_b,t_a) = e^{-iH(t_b-t_N)}\, e^{-iH(t_N-t_{N-1})} \cdots\cdots e^{-iH(t_1-t_0)}$

  (5.) Insert I ... $\int \ldots$

$$I = \int dq_n |q_n\rangle\langle q_n|$$

⑥ $K(q_b,t_b;q_a,t_a) = \int dq_n, dq_{n-1} \cdots dq_1$

$\langle q_b| e^{-iH(t_b-t_N)}|q_N\rangle\langle q_N|e^{-iH(t_N-t_{N-1})}|q_{N-1}\rangle \cdots$

$\cdots \langle q_1|e^{-iH(t_1-t_a)}|q_a\rangle$

⑦ For small ε - intervals

$\langle q_n|e^{-iH(t_n-t_{n-1})}|q_{n-1}\rangle \simeq \langle q_n|1-i\varepsilon H|q_{n-1}\rangle + \cdots$

$= \underset{ⓐ}{\delta(q_n - q_{n-1})} - i\varepsilon \underset{ⓑ}{\langle q_n|H|q_{n-1}\rangle}$

8)

a) $\delta(q_n - q_{n-1}) = \frac{1}{2\pi}\int e^{iP(q_n-q_{n-1})} dP$

b) $\langle q_n|P^2|q_{n-1}\rangle = \frac{1}{2\pi}\int e^{iP(q_n-q_{n-1})} P^2 dP$

$\psi(x) = \langle x|\psi\rangle$ $\quad \langle x|P^2|\psi\rangle = -\nabla^2|\psi(x)\rangle$ / $\underline{\text{here } x \equiv q_n}$

Therefore

$\langle q_n|P^2|q_{n-1}\rangle = -\nabla^2\langle q_n|q_{n-1}\rangle = \frac{-\nabla^2}{2\pi}\int e^{iP(q_n-q_{n-1})} dP =$

$= \frac{1}{2\pi}\int e^{iP(q_n-q_{n-1})} P^2 dP$

c) $\langle q_n|V(q)|q_{n-1}\rangle = V(q)\langle q_n|q_{n-1}\rangle = \frac{1}{2\pi}\int e^{iP(q_n-q_{n-1})} V(q_n)\, dP$

9) If the Hamiltonian has the form

$$H = \frac{P^2}{2m} + V(q)$$

$\langle q_n|e^{-iH(t_n-t_{n-1})}|q_{n-1}\rangle = \frac{1}{2\pi}\int e^{iP(q_n-q_{n-1})}[1 - i\varepsilon H(P,q_n)]\, dP$

$\approx \frac{1}{2\pi}\int e^{iP(q_n-q_{n-1})} e^{-i\varepsilon H(P,q_n)} dP = \frac{1}{2\pi}\int e^{i\varepsilon[P\dot{q}_n - H(P,q_n)]} dP$

$\dot{q}_n = (q_n - q_{n-1})/\varepsilon$

10) Propagator itself

$\sum^{N+1} \cdots$

$$K(q_b,t_b;q_a,t_a) \int \prod_{n=1}^{N} dq_n \prod_{n=1}^{N+1} \frac{dp_n}{2\pi} e^{i\varepsilon \sum_{n=1} (p_n \dot{q} - H(p_n,q_n))}$$

ii) $N \to \infty$ and $\varepsilon \to 0$ such that $\varepsilon(N+1) \to t_b - t_a$

$$K(q_b,t_b;\ q_a t_a) \int Dq\, DP \exp\left[i\int_{t_a}^{t_b} (P\dot{q} - H(P,q))\, dt\right]$$

or

$$K(q_b,t_b;\ q_a,t_a) \int Dq\, Dp\, \theta(t_b-t_a) \exp\left[i\int_{t_a}^{t_b} (p\dot{q} - H(p,q))dt\right] \times$$
$$\times \exp\left[-\eta(t_b-t_a)\right]$$

Feynman Path integral for the Propagator

- Functional Integral
  takes $q(t)$ an $P(t)$ and makes an ordinary number

- $Dq = \prod_{n=1}^{N} dq_n(t)$

- $Dp = \prod_{n=1}^{N+1} \frac{dp_n}{2\pi}(t)$

- p and q - are ordinary numbers
  no operators anymore

- $H(p,q)$ - is an ordinary function

- There is no ordering of symbols
  - if $H = \alpha\hat{p}^2 + \beta q^2$ - result is the same
  - if $\hat{p}\hat{q}$ terms move $\hat{q}\hat{p}$

- Continue the discussion for $H = \frac{p^2}{2m} + V(q)$)

a) Lets study the part

$$\int dp_n\, e^{i\varepsilon p_n \dot q - i\varepsilon \frac{p_n^2}{2m}} = //$$

$$\int_{-\infty}^{+\infty} e^{-(at^2+bt+c)}\, dt = \sqrt{\frac{\pi}{a}}\, e^{\left(\frac{b^2-4ac}{4a}\right)}$$

$$\int_{-\infty}^{\infty} e^{-t^2}\, dt = \sqrt{\pi}$$

$$// = e^{-i\varepsilon\left(-p_n\dot q + \frac{p_n^2}{2m} + \frac{m\dot q^2}{2} - \frac{m\dot q^2}{2}\right)}$$

$$= e^{-\left(-i\varepsilon p_n \dot q + i\varepsilon\frac{p_n^2}{2m} - i\varepsilon\frac{m\dot q^2}{2} + i\varepsilon\frac{m\dot q^2}{2}\right)}$$

$$\frac{\sqrt{i\varepsilon}\, p_n}{\sqrt{2m}} \qquad \frac{\sqrt{i\varepsilon}\sqrt{m}\,\dot q}{\sqrt 2}$$

$$= -\left(\frac{\sqrt{i\varepsilon}\,p_n}{\sqrt{2m}} - \frac{\sqrt{i\varepsilon}\,\dot q\sqrt m}{\sqrt 2}\right)^2 \cdot e^{i\varepsilon\frac{m}{2}\dot q^2}$$

$$\cdot\, \frac{dp_n\sqrt{2m}}{\sqrt{i\varepsilon}} \qquad \frac{\sqrt{i\varepsilon}}{\sqrt{2m}}$$

$$\int dp_n\, e^{i\varepsilon p_n\dot q - i\varepsilon\frac{p_n^2}{2m}} = \int e^{-x^2}dx\, \frac{\sqrt{2m}}{\sqrt{i\varepsilon}}\, e^{+i\frac{m}{2}\dot q^2} = \frac{\sqrt{2m\pi}}{\sqrt{i\varepsilon}}\, e^{\frac{im}{2}\dot q^2}$$

– using above integral one obtains

$$K(q_b,t_b;q_a,t_a) = \underbrace{\lim_{N\to\infty}\left(\frac{m}{2\pi i\varepsilon}\right)^{\frac{N+1}{2}}}_{C} \int \prod_{n=1}^{N} dq_n\, e^{i\varepsilon\sum_{n=1}^{N+1}\left(\frac{m\dot q_n^2}{2} - V(q_n)\right)}$$

$$K(q_b,t_b;q_a,t_a) = \int Dq\, e^{i\int_{t_a}^{t_b} L(q,\dot q)\,dt} =$$

$$= \int Dq\, e^{iS(q)}$$

$\Rightarrow$ Can be Generalized for any $\hat{H}$