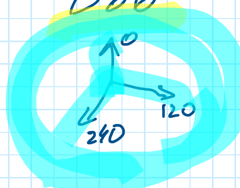
QC13

Monday, March 22, 2021 6:24 PM

Quantum Gates and Circuits

Alice → Qubits → Bob



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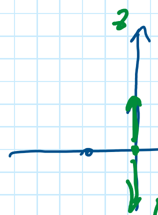


$\begin{pmatrix}1\\0\end{pmatrix} = |\uparrow\rangle = |0\rangle$

$\begin{pmatrix}0\\1\end{pmatrix} = |\downarrow\rangle = |1\rangle$

$|\psi\rangle = \alpha|\uparrow\rangle + \beta|\downarrow\rangle$

$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

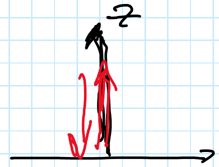
Alice → Qubits → Bob

Modification – Manipulation

Quantum Gate

Qubits

– ordered – standard basis



$$\left(\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right) \equiv (|\uparrow\rangle, |\downarrow\rangle) = (|0\rangle, |1\rangle)$$

$\boxed{|0\rangle \equiv \begin{bmatrix}1\\0\end{bmatrix}}$ $|0\rangle$

$\boxed{|1\rangle \equiv \begin{bmatrix}0\\1\end{bmatrix}}$ $|1\rangle$

– Arbitrary Qubit $|\psi\rangle = a_0|0\rangle + a_1|1\rangle$

$|a_0|^2 + |a_1|^2 = 1$

$|a_0|^2$, $|a_0|^2 + |a_1|^2 = 1$

⟹ Two qubit system

$\otimes$ $\quad \begin{bmatrix}1\\0\end{bmatrix}\otimes\begin{bmatrix}1\\0\end{bmatrix}, \quad \begin{bmatrix}1\\0\end{bmatrix}\otimes\begin{bmatrix}0\\1\end{bmatrix}, \quad \begin{bmatrix}0\\1\end{bmatrix}\otimes\begin{bmatrix}1\\0\end{bmatrix}, \quad \begin{bmatrix}0\\1\end{bmatrix}\otimes\begin{bmatrix}0\\1\end{bmatrix}$

$|0\rangle\otimes|0\rangle, \quad |0\rangle\otimes|1\rangle, \quad |1\rangle\otimes|0\rangle, \quad |1\rangle\otimes|1\rangle$

$|00\rangle \quad \otimes \cdots \quad |01\rangle \quad |10\rangle \quad |11\rangle$

2^n Base states for 2 qubits

$|\lambda_i\rangle$ (4 states) $\quad \boxed{\langle\lambda_i|\lambda_j\rangle = \delta_{ij}}$

$i, j = 1, 4$

The CNOT Gate

2-inputs ⟹ 2-output

$0 \to |0\rangle \quad 1 \to |1\rangle$

Input x	Input y	output x	output $x\oplus y$
0	0	0	0
0	1	0	1
1	0	1	1
1	1	1	0

Input x	Input y	Output x	Output $x\oplus y$
$\lvert 0\rangle$	$\lvert 0\rangle$	$\lvert 0\rangle$	$\lvert 0\rangle$
$\lvert 0\rangle$	$\lvert 1\rangle$	$\lvert 0\rangle$	$\lvert 1\rangle$
$\lvert 1\rangle$	$\lvert 0\rangle$	$\lvert 1\rangle$	$\lvert 1\rangle$
$\lvert 1\rangle$	$\lvert 1\rangle$	$\lvert 1\rangle$	$\lvert 0\rangle$

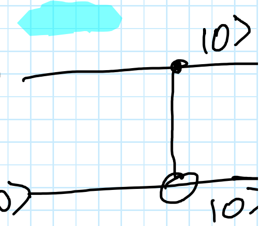
CNOT

Input	Output
$\lvert 00\rangle$	$\lvert 00\rangle$
$\lvert 01\rangle$	$\lvert 01\rangle$
$\lvert 10\rangle$	$\lvert 11\rangle$
$\lvert 11\rangle$	$\lvert 10\rangle$

Extend it natural way
$0 \to |0\rangle \quad 1 \to |1\rangle$

$|1\rangle \; |1\rangle \; |0\rangle \; |0\rangle$

$|0\rangle \; |0\rangle \; |1\rangle \; |0\rangle$



basis states

– One Qubit $\quad |\psi\rangle = a_0|0\rangle + a_1|1\rangle$

$|\psi\rangle = \alpha|00\rangle + \beta|01\rangle + \cdots$

$\gamma|10\rangle + \delta|11\rangle$

- Two qubits

$$|\psi\rangle = a_{00}|00\rangle + a_{01}|01\rangle + a_{10}|10\rangle + a_{11}|11\rangle$$

$$|a_{00}|^2 + |a_{10}|^2 + |a_{01}|^2 + |a_{11}|^2 = 1$$

$$\to \quad CNOT|\psi\rangle = CNOT\left(a_{00}|00\rangle + a_{01}|01\rangle + a_{10}|10\rangle + a_{11}|11\rangle\right) =$$

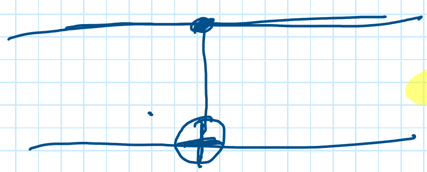
$$= a_{00}|00\rangle + a_{01}|01\rangle + a_{10}|11\rangle + a_{11}|10\rangle = |\psi\rangle'$$

Matrix

$CNOT = ?$

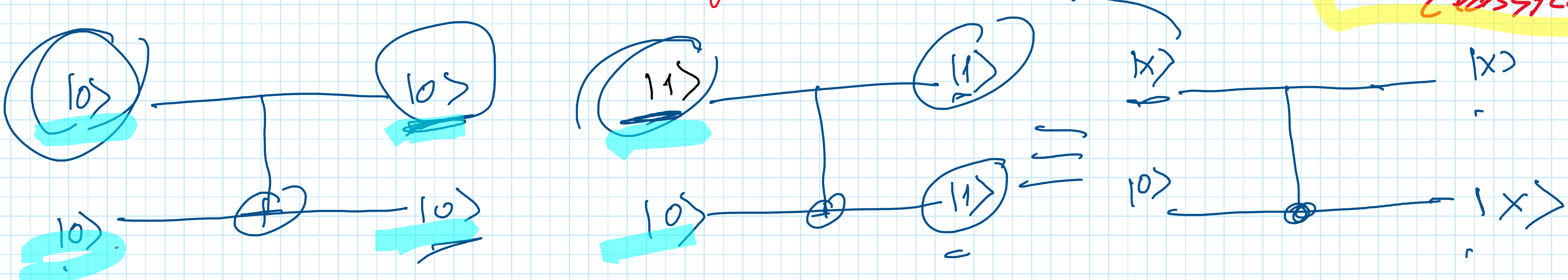
flips the probabi[lity] amplitude

CNOT



- Consider CNOT for basis qubits

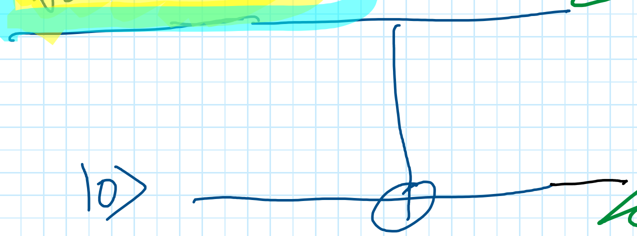
similar to classical



Consider CNOT for interference bits

$\frac{1}{\sqrt{2}}|0\rangle+\frac{1}{\sqrt{2}}|1\rangle$

$|0\rangle$



$\frac{1}{\sqrt{2}}|0\rangle|0\rangle+\frac{1}{\sqrt{2}}|1\rangle|1\rangle$ — Can not seperate

it makes Entan

$$CNOT\left(\frac{1}{\sqrt{2}}|0\rangle+\frac{1}{\sqrt{2}}|1\rangle\right)\otimes|0\rangle = CNOT\left(\frac{1}{\sqrt{2}}\overset{|00\rangle}{|0\rangle|0\rangle}+\frac{1}{\sqrt{2}}\overset{|10\rangle}{|1\rangle|0\rangle}\right) = \frac{1}{\sqrt{2}}|00\rangle$$

Quantum Gates: G — Permuting orthogonal state

Thus $\hat{G}$ - is orthogonal matrix

$G^\dagger G = I$

$$\hat{G}\,|\psi_n\rangle = |\tilde{\psi}_n\rangle$$

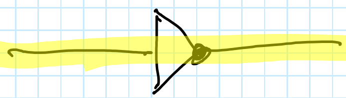
$\hat{G}_s$ - mathematically are Matrices

— There are infinite number of Quantum Gates

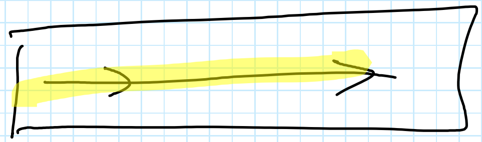
Quantum Gates Acting on One Qubit

– Classical Gates

Not Gate.



Identity Gate



Not Gate	
Inputs	Output
→ 0	→ 1
→ 1	→ 0

Identity Gate	
Input	Output
→ 0	→ 0
→ 1	→ 1

C		Q
0	→	$\|0\rangle$
1	→	$\|1\rangle$

⇒ Quantum Identity Gate I

Input	Output
→ $\|0\rangle$	→ $\|0\rangle$
→ $\|1\rangle$	→ $\|1\rangle$

$$\begin{cases} \hat{I}\,|0\rangle = |0\rangle \\ \hat{I}\,|1\rangle = |1\rangle \end{cases} \Rightarrow \begin{aligned} \hat{I}\begin{pmatrix}1\\0\end{pmatrix} &= \begin{pmatrix}1\\0\end{pmatrix} \\ \hat{I}\begin{pmatrix}0\\1\end{pmatrix} &= \begin{pmatrix}0\\1\end{pmatrix} \end{aligned}$$

$\hat{I} =$

$\hat{I} =$

$I(\alpha|0\rangle + \beta|1\rangle) = \alpha|0\rangle + \beta|1\rangle \longrightarrow$

$\rightarrow I(\alpha|0\rangle + \beta|1\rangle) = \alpha|0\rangle \ldots$

Other Possibility

$\boxed{\hat{Z} \text{ gate}}$

$$\left\{ \begin{array}{l} \hat{Z}|0\rangle = |0\rangle \\ \hat{Z}|1\rangle = -|1\rangle \end{array} \right. \qquad \hat{Z}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} \qquad \hat{Z}\begin{pmatrix}0\\1\end{pmatrix} = -\begin{pmatrix}0\\1\end{pmatrix}$$

$$\hat{Z}(\alpha|0\rangle + \beta|1\rangle) = \alpha\hat{Z}|0\rangle \ldots$$

Preserves the Basis v...

Quantum NOT Gate

Input	Out
$\|0\rangle$	$\|1\rangle$
$\|1\rangle$	$\|0\rangle$

$$\hat{X}|0\rangle = |1\rangle \qquad \hat{X}\begin{pmatrix}1\\0\end{pmatrix} = (\ldots$$

$$\hat{X}|1\rangle = |0\rangle \qquad \hat{X}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$$

– Other Possibility

InP	Out
$\|0\rangle$	$-\|1\rangle$
$\|1\rangle$	$\|0\rangle$

$$\hat{y}|0\rangle = -|1\rangle \qquad \hat{y}\begin{pmatrix}1\\0\end{pmatrix} = -\begin{pmatrix}0\\1\end{pmatrix}$$

$$\hat{y}|1\rangle = |0\rangle \qquad \hat{y}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$$

Pauli Gates

$\Rightarrow$ Infinite Number of one input $\rightarrow$ one o
Gates

The Hadamard Gate.

$$H = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} = \boxed{\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}}$$

$$\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1$$

$$\hat{H}|0\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\hat{H}|1\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \end{bmatrix} - \frac{1}{\sqrt{2}}\Big[$$

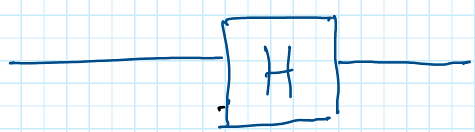
$$\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|$$

— This Gate used to put standard Basis into superposition

$$\hat{H}|0\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) \qquad H|1\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right.$$

$$H\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{\sqrt{2}}\left(\begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix}\right) = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \qquad H\begin{pmatrix}0\\1\end{pmatrix} = \frac{1}{\sqrt{2}}\left[\begin{pmatrix}1\\0\end{pmatrix} + \right.$$

$$\boxed{\hat{H} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}}$$



Acting