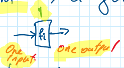
# QC18

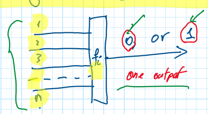
Monday, April 12, 2021 4:11 PM

## The Deutsch-Jozsa Algorithm

- Deutches algorithm → a function of one variable




- Consider a function of n-variables



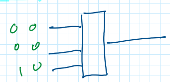

Constant — output always 1, always 0

Balanced — output half 1, half 0

- if these functions given at random
  how many evaluations (to call) of these functions we need to define — constant / Balanced

- Example $n=3$ $2^3$ inputs

(0,0,0)
(0,0,1)
(0,1,0)
(0,1,1)
(1,0,0)
(1,0,1)
(1,1,0)
(1,1,1)




- Classically - suppose we evaluate

1st { $f(0,0,0) = 1$ — can not say anything

2nd { next $f(0,0,1) = 0$ — Balanced!
but if $f(0,0,1) = 1$ — can not say anything

3rd { $f(0,1,0) = 1$ — no

4th { $f(0,1,1) = 1$ — no

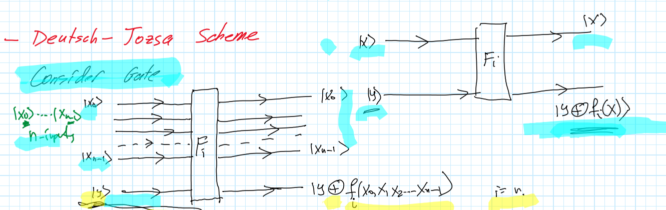
5th { $f(1,1,0) = 1$ — Constant!

min 2 measurements, max $5 = 2^{n-1}+1$

- for $n >> 1$ $2^{n-1}+1 \sim e^n$ — impossible number of classical measurements

- Deutsch-Jozsa Scheme

Consider Gate

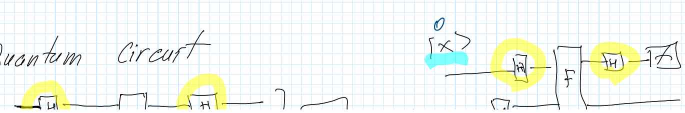


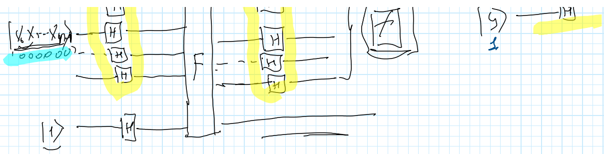

- check $|x_0, x_1 \cdots x_{n-1}\rangle \otimes |0\rangle \longrightarrow |x_0, x_1 \cdots x_{n-1}\rangle \otimes |0 \oplus f(x_0 \cdots x_{n-1})\rangle = |x_0, x_1 \cdots x_{n-1}\rangle \otimes |f_i(x_0 \cdots x_{n-1})\rangle$

$|x_0, x_1 \cdots x_{n-1}\rangle \times |1\rangle \longrightarrow |x_0, x_1 \cdots x_{n-1}\rangle \otimes |1 \oplus f(x_0 \cdots x_{n-1})\rangle = |x_0, x_1 \cdots x_{n-1}\rangle \otimes |f(x_0 \cdots x_{n-1}) \oplus 1\rangle$

!!!

- Quantum Circuit






Doetake-

# The Kronecker Product of Hadamard Matrices

$\Rightarrow$ Matrix for Hadamard gate is given

$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

This follows from

$\hat{H} = ?$

$H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

$H|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$

$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} =$

$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{1}{\sqrt{}}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\frac{1}{\sqrt{}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$

$\Rightarrow$ Suppose we input two qubits and send throug H gates

$H(|0\rangle|0\rangle) =$ $|0\rangle$ –H– $\;|0\rangle$ –H–

H( )

H( )

H( )

$\Rightarrow$ We can rewrite this in terms of $\boxed{\text{4-d kets}}$

Example $\quad |0\rangle \otimes |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

$|0\rangle \otimes |1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

$|1\rangle \otimes |0\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$

$|1\rangle \otimes |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

4d

$H|0\rangle \otimes |0\rangle = H_4^{\otimes 2}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}(|00\rangle + |01\rangle + |10\rangle + |11\rangle) = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

$H|0\rangle \otimes |1\rangle = H_4^{\times(2)}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}(|00\rangle - |01\rangle + |10\rangle - |11\rangle) = \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$

$H|1\rangle \otimes |0\rangle = H^{\otimes 2}\begin{pmatrix} 0 \\ 0 \\ \dots \end{pmatrix} = \; - \; - \; - \; - \; - \; - \quad \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ -1 \\ \dots \end{bmatrix}$

$$H|1\rangle \otimes |1\rangle = H^{\otimes 2}_{4}\begin{bmatrix}0\\0\\0\\1\end{bmatrix} = \cdots\cdots \frac{1}{2}\begin{bmatrix}1\\-1\\-1\\0\end{bmatrix}$$

— Solve for H in 4d.

⇒ For 3-qubit system

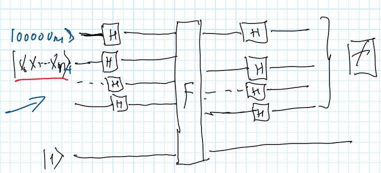
⇒ For n-qubit system

Recursive Formula
Kronecker Product

⇒ Each row is proportional to $\left(\frac{1}{\sqrt{2}}\right)^n$

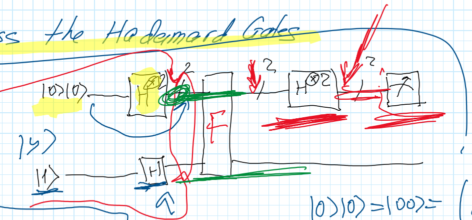
Same for the state

⇒ Return to Deutsch-Jozsa Algorithm



⟹ Consider case of $n=2$

Step 1. The Qubits Pass the Hadamard Gates




$|0\rangle|0\rangle = |00\rangle = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$

– |y⟩ enter |1⟩ $\quad H|1\rangle = \boxed{\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle}$

(The above calculation works for every n
After n-qubits have passed $H^{\otimes n}$ they are
in superposition of all possible states
each having same probability amplitude $\left(\frac{1}{\sqrt{2}}\right)^n$

Step 1

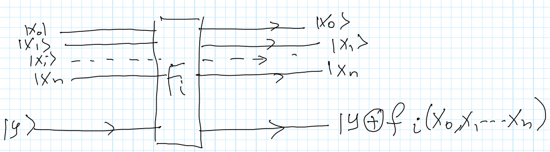
– So for $n=2$ we have after $H^{\otimes 2}$ and $H$ (for y)

Input after Hadamards

$$\frac{1}{2}\left(|00\rangle + |01\rangle + |10\rangle + |11\rangle\right) \otimes \left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right) =$$

$\hat{F}_i$ } Entering To $F_i$ Gate

Step 2: Remember the Gate

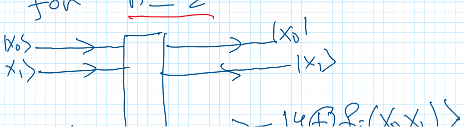



Input $\qquad\qquad$ Output

$|x_0\rangle|x_1\rangle\cdots|x_n\rangle \otimes |0\rangle \longrightarrow |x_0\rangle|x_1\rangle\cdots|x_n\rangle \otimes |f_i(x_1\cdots x_n)\rangle$

$|x_0\rangle|x_1\rangle\cdots|x_n\rangle \otimes |1\rangle \longrightarrow |x_0\rangle|x_1\rangle\cdots|x_n\rangle \otimes |f_i(x_1\cdots x_n) \oplus 1\rangle$

for $n=2$




input

$|x_1\rangle|x_2\rangle \otimes |0\rangle \rightarrow |x_0\rangle|x_1\rangle \otimes |f_i(x_1 x_2)\rangle$

$|x_1\rangle|x_2\rangle \otimes |1\rangle \rightarrow |x_0\rangle|x_1\rangle \otimes |f_i(x_1 x_2) \oplus 1\rangle$ } !!

Using this

Using the fact that if $a$ is either $0$ or $1$

$$|a\rangle - |a \oplus 1\rangle = (-1)^a(|0\rangle - |1\rangle)$$

$a \to f_i$

$\Rightarrow$

One obtains

(a) –
(b) –
(c) –
(d) –

– Bottom qubit is not entangled with the top

$$(\qquad\qquad) \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

!

$n=2$

(The above argument works for any $n$ in which

$$\left(\sum \frac{1}{(\sqrt{2})^n}(-1)^{f(x_0,x_1,\dots,x_{n-1})}|x_0 \dots x_{n-1}\rangle\right) \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

all possible $x_0, x_1 \dots x_n$ combinations

Step 3 The Top Qubits Pass through $H^{\otimes 2}$ Gate

$$\sim \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \begin{pmatrix}0\\0\\0\\1\end{pmatrix} = \begin{pmatrix}(-1)^{f(0,0)}\\(-1)^{f(0,1)}\\(-1)^{f(1,0)}\\(-1)^{f(1,1)}\end{pmatrix}; \quad (*)$$

$$|00\rangle = \begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$$

$$|01\rangle = \begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$$

$$|10\rangle = \begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}$$

$|11\rangle = \dots$

$|11\rangle = \begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$ $\quad \otimes 2$

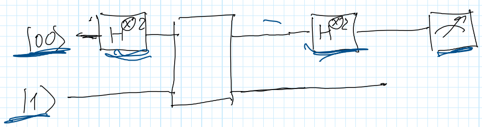
Passing this through Hadamard Gate

$$H^{\otimes 2}\begin{bmatrix}(-1)^{f(00)}\\(-1)^{f(01)}\\(-1)^{f(10)}\\(-1)^{f(11)}\end{bmatrix} = \boxed{\phantom{xx}}\boxed{\phantom{xx}} = \text{Eq. 4}\begin{pmatrix}+\\ \cdot\\ \\ \end{pmatrix}$$

$|00\rangle$ $\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$

- Lets check the Top entry

⇒ For The Deutsch – Jozsa Algorithm we consider




– This correspond to a top state in Eq(4) which results of the output

$$\begin{pmatrix}
\frac{1}{4}\left((-1)^{f(0,0)} + (-1)^{f(0,1)} + (-1)^{f(1,0)} + (1)^{f(1,1)}\right)\\
\frac{1}{4}\left((-1)^{f(0,0)} - (-1)^{f(0,1)} + (-1)^{f(1,0)} - (1)^{f(1,1)}\right)\\
\frac{1}{4}\left((-1)^{f(0,0)} + (-1)^{f(0,1)} - (-1)^{f(1,0)} - (-1)^{f(1,1)}\right)\\
\frac{1}{4}\left((-1)^{f(0,0)} - (-1)^{f(0,1)} - (-1)^{f(1,0)} + (-1)^{f(1,1)}\right)
\end{pmatrix}$$

– This is qubit which is being measured at the Top of output

Consider (*)

– if $f_i$ – const function $\boxed{a}$

$f(0,0) = 0$
$f(0,1) = 0$
$f(1,0) = 0$
$f(1,1) = 0$

or $\boxed{b}$

$f(0,0) = 1$
$f(0,1) = 1$
$f(1,0) = 1$
$f(1,1) = 1$

Balance

| | |
|---|---|
| 1 | 1 |
| 1 | 0 |
| 0 | 0 |
| 0 | 1 |

$\boxed{a}$

(5) $\frac{1}{4}\begin{pmatrix} \cdot \\ 0 \\ 0 \\ 0 \end{pmatrix} \leftarrow$ (6) $\frac{1}{4}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \leftarrow$ $|00\rangle = \frac{1}{4}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

– if $f_1$ – balanced $f(0,0)=0$ $f(1,0)=1$ $f(0,1)=0$ $f(1,1)=1$

$\frac{1}{4}\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$

$\frac{1}{4}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

$\frac{1}{4}\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

Step 4. Measure of Top Qubits

– if functions are constant Top Qubit gives $|00\rangle$

– Otherwise if balanced Top qubit gives $|01\rangle$ $|10\rangle$ $|11\rangle$

– Same is true if $F_{i=n}$ – functions are const

Top Qubit gives $|0000\cdots0_n\rangle$ – const.

if Balanced $|001_i\cdots0\rangle$ $[i=1\cdots n]$

– Deutsch-Jozsa algorithm needs only one measurement instead of $2^{n-1}+1$.