QC2

Wednesday, January 29, 2025 4:10 PM

Spin and Photon Polarization

⇒ In General we discuss Rotation in QM by acting Rotational Operator On Quantum State

$$\hat{R}(\hat{n},\theta)|\psi\rangle = |\psi'\rangle$$

– Using Active and Passive Transformation Paradigm we can state that Rotating Quantum state around $\hat{n}$ by θ is equivalent to rotating our Reference Frame–Measuring apparatus by $-\theta$ around $\hat{n}$

Spin

⇒ Consider now the Quantum state to be spinor with spin $\frac{1}{2}$

– Eigenstate of spin state along and opposite to $\hat{z}$ axis is

$$|\uparrow\rangle = \begin{pmatrix}1\\0\end{pmatrix} \quad \text{and} \quad |\downarrow\rangle = \begin{pmatrix}0\\1\end{pmatrix}$$

We Can consider them as a orthonormal basis state

$$(|\uparrow\rangle, |\downarrow\rangle) =$$

– Rotating this state by θ around $\hat{n}$ is defined by the rotation operator

$$R(n,\vartheta) \equiv D_{\frac{1}{2}}(n,\vartheta) = I\cos\left(\frac{\vartheta}{2}\right) - i(\vec{\sigma}\cdot\vec{n})\sin\frac{\vartheta}{2}$$

where $\vec{\sigma}\cdot\vec{n} = \sigma_x n_x + \sigma_y n_y + \sigma_z n_z$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

- If we would like to rotate the reference frame then $\vartheta = -\theta$

- Consider that original we measured the quantum state of spin-$\frac{1}{2}$ particle $|\gamma\rangle$ in z direction

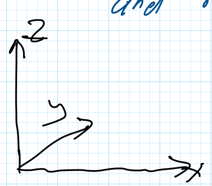
- Befor the measurement
  $|\gamma\rangle = \alpha|\uparrow\rangle + \beta|\downarrow\rangle$

- α, β probability amplitude that out come of measurement is along or opposite to $\hat{z}$
  - Naturally $\alpha^2 = \beta^2 = \frac{1}{2}$

1. Suppose the measurement resulted in along $\hat{z}$ state
2. Thus state will collapse to $|\uparrow\rangle$ state
3. Now you want to measure this collapsed state in arbitrary θ direction

You take this original basis states and rotate by $-\theta$ along $\hat{y}$ axis

x y z
y z x



$$D(y,-\theta) \equiv R(y,-\theta) = I\cos\frac{\theta}{2} + i\sigma_y\sin\frac{\theta}{2} = \begin{pmatrix} \cos\theta/2 & \sin\theta/2 \\ -\sin\theta/2 & \cos\theta/2 \end{pmatrix}$$

$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

$$D(\hat{y},-\theta)\left|\uparrow\right\rangle = \begin{pmatrix} \cos\theta/2 & \sin\theta/2 \\ -\sin\theta/2 & \cos\theta/2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\theta/2 \\ -\sin\theta/2 \end{pmatrix} \equiv \left|\nearrow_\theta\right\rangle$$

$$D(\hat{y},-\theta)\left|\downarrow\right\rangle = \begin{pmatrix} \cos\theta/2 & \sin\theta/2 \\ -\sin\theta/2 & \cos\theta/2 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \sin\theta/2 \\ \cos\theta/2 \end{pmatrix} = \left|\swarrow_\theta\right\rangle$$

Thus new orthonormal basis is $\left(\left|\nearrow_\theta\right\rangle, \left|\swarrow_\theta\right\rangle\right)$

4) Calculate probability that collapsed state $\left|\psi\right\rangle$ will be measured along $\left|\nearrow_\theta\right\rangle$ and $\left|\swarrow_\theta\right\rangle$

$$\begin{pmatrix} \left\langle\nearrow_\theta\right| \\ \left\langle\nearrow_\theta\right| \end{pmatrix}\left|\uparrow\right\rangle = \begin{pmatrix} \left\langle\nearrow_\theta|\uparrow\right\rangle \\ \left\langle\swarrow_\theta|\uparrow\right\rangle \end{pmatrix}$$

$$\left\langle\nearrow_\theta|\uparrow\right\rangle = \left(\cos\theta/2, -\sin\theta/2\right)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\theta/2 \\ 0 \end{pmatrix}$$

$$\left\langle\swarrow_\theta|\uparrow\right\rangle = \left(\sin\theta/2 \;\; \cos\theta/2\right)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \sin\theta/2 \\ 0 \end{pmatrix}$$

Algebra for Spin

$\Rightarrow$ General Property of Rotation

$\Rightarrow$ Rotating Spin states

$$T = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\frac{1}{\sqrt{2}}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

$(|\uparrow\rangle, |\downarrow\rangle)$ $\quad |\uparrow\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\quad |\downarrow\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$(|\rightarrow\rangle\ |\leftarrow\rangle)$ $\quad |\rightarrow\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$ $\quad |\leftarrow\rangle \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$

$$A^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$A^\dagger|\uparrow\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

Measuring in vertical direction

$$\hat{S}|\rightarrow\rangle = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} = \frac{1}{\sqrt{2}}|\uparrow\rangle - \frac{1}{\sqrt{2}}|\downarrow\rangle$$

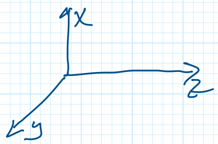
$$|\leftarrow\rangle = \frac{1}{\sqrt{2}}|\uparrow\rangle + \frac{1}{\sqrt{2}}|\downarrow\rangle$$

$\Rightarrow$ Equivalent ~~States~~

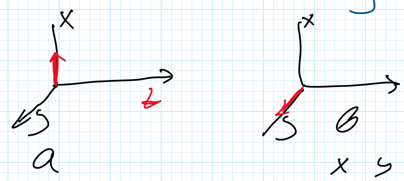
$\Rightarrow$ Rotating Apparatus in 60°

$$\left(\begin{bmatrix} \sqrt{3}/2 \\ -\frac{1}{2} \end{bmatrix}, \begin{bmatrix} 1/2 \\ \sqrt{3}/2 \end{bmatrix}\right)$$

⇒ Consider Photon Polarization



two polarization direction x and y



$$x \quad y \quad z \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

⇒ Quantum state of (a) $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

Quantum state (b) $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

We can drop z

$$\begin{pmatrix} 1 & 0 \end{pmatrix}$$
$$\begin{pmatrix} 0 & 1 \end{pmatrix}$$

⇒ Rotation in this case is defined by Rotation of a vector with unit x, and y components

– in our case it is a rotation around z-axis

$$-R_z(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$R_z(\theta)|\psi\rangle = |\psi'\rangle$$

– If we are rotating the reference Frame –

$$R_z(-\theta) = \begin{pmatrix} \cos\theta & +\sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

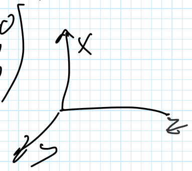
(x_x)

$$R_z(-\theta)\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}\cos\theta & \sin\theta & 0\\ -\sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}\cos\theta\\ -\sin\theta\\ 0\end{pmatrix}$$

$$\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & -i\\ 0 & i & 0\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}$$

$$R_z(-\theta)\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}\cos\theta & \sin\theta & 0\\ -\sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}\sin\theta\\ \cos\theta\\ 0\end{pmatrix}$$

$\Rightarrow$ Suppose initially you have a photon
state in $|x\rangle = \begin{pmatrix}1\\0\\0\end{pmatrix}$ $|y\rangle = \begin{pmatrix}0\\1\\0\end{pmatrix}$



(1) $|\psi\rangle_\gamma = \alpha_x|x\rangle + \alpha_y|y\rangle$

(2) You measured and it came up in $\hat{x}$
collapsed to $|x_0\rangle$

3) You now measure in $\hat{y}$ and $-\hat{x}$ direction
$\theta = 30^\circ$

New Basis is $|\hat{y}_{30}\rangle = \begin{pmatrix}0\\-1\\0\end{pmatrix}$ $|-x\rangle = \begin{pmatrix}1\\0\\0\end{pmatrix}$

~~Let~~ what is the probability to meas in
y-direction $|\hat{y}_{30}\rangle$ and in $|x\rangle$
finally.

$$|X_0\rangle = \beta_1 |y_{s0}\rangle + \beta_2 |-x\rangle$$

$$A = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \beta_1 \\ \beta_2 \\ 0 \end{pmatrix}$$

$$\boxed{\beta_1 = 0 \qquad \beta_2 = 1}$$

$\Rightarrow$ Do the same if $\theta = 45^\circ$

$\Rightarrow$ Then again rotate to $\theta = 45\%$