QC5

Monday, February 15, 2021 3:47 PM

Entangled Qubits

- Remember Definition

$$|\psi_{AB}\rangle = \gamma_{00}|\lambda_{00}\rangle + \gamma_{01}|\lambda_{01}\rangle + \gamma_{10}|\lambda_{10}\rangle + \gamma_{11}|\lambda_{11}\rangle$$

① if $\gamma_{00}\cdot\gamma_{11} = \gamma_{01}\cdot\gamma_{10}$ — not entangled

② if $\gamma_{00}\cdot\gamma_{11} \neq \gamma_{01}\cdot\gamma_{10}$ — entangled

- Consider

$$|\psi_{AB}\rangle = \underbrace{\tfrac{1}{2}}_{\gamma_{00}}|a_0\rangle|b_0\rangle + \underbrace{\tfrac{1}{2}}_{\gamma_{01}}|a_0\rangle|b_1\rangle + \underbrace{\tfrac{1}{\sqrt{2}}}_{\gamma_{10}}|a_1\rangle|b_0\rangle + \underbrace{0}_{\gamma}|a_1\rangle|b_1\rangle \quad (*)$$

ⓐ $\gamma_{00}\gamma_{11} = 0 \neq \gamma_{01}\gamma_{10} = \frac{1}{2\sqrt{2}}$

ⓑ $\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + 0^2 = 1$

$$\boxed{\gamma_{00}^2 + \gamma_{01}^2 + \gamma_{10}^2 + \gamma_{11}^2 = 1}$$

$\langle\lambda_n|\lambda_n\rangle = \delta_{nn}$

$|\psi\rangle = \sum \lambda_n |\lambda_n\rangle$ $\quad \langle\lambda_n|\lambda_n\rangle = 1$

- what happens if one of them makes a measurement

Alice Measures first $\quad |\psi_{AB}\rangle = |a_0\rangle \left(\frac{1}{2}|b_0\rangle + \frac{1}{2}|b_1\rangle\right) + |a_1\rangle\left(\frac{1}{\sqrt{2}}|b_0\rangle + 0|b_1\rangle\right)$

|Basis⟩ $\quad$ Number · |Basis⟩ $\quad$ $\frac{\sqrt{2}}{2}$ $\quad$ |Basis⟩ · $\frac{1}{\sqrt{2}}$

$\langle \text{Basis}|\text{Basis}\rangle = 1$

$$\left(\tfrac{1}{2}\langle b_0| + \tfrac{1}{2}\langle b_1|\right)\left(\tfrac{1}{2}|b_0\rangle + \tfrac{1}{2}|b_1\rangle\right)$$

$$\frac{1}{4} + \frac{1}{4} = \frac{1}{2} \quad \sqrt{\tfrac{1}{2}}$$

Alices Perspectives

$$|\psi_{AB}\rangle = \frac{1}{\sqrt{2}}|a_0\rangle\left(\frac{1}{\sqrt{2}}|b_0\rangle + \frac{1}{\sqrt{2}}|b_1\rangle\right) + \frac{1}{\sqrt{2}}|a_1\rangle\left(1|b_0\rangle + 0|b_1\rangle\right)$$

$|\langle a_0|\psi_{AB}\rangle|^2 = \frac{1}{2}$ $\quad \boxed{0}$ $\quad |a_0\rangle$ ⓐ

$|\langle a_1|\psi_{AB}\rangle|^2 = \frac{1}{2}$ $\quad \boxed{1}$ $\quad |a_1\rangle$ ⓑ

1

$$|\tilde{\psi}_{AB}\rangle = |a_0\rangle\left(\frac{1}{\sqrt{2}}|b_0\rangle + \frac{1}{\sqrt{2}}|b_1\rangle\right)$$

- Bob does the measurement

ⓐ $|\langle b_0|\tilde{\psi}_{AB}\rangle|^2 = \boxed{\frac{1}{2}}$ (0) $\qquad |\langle b_1|\tilde{\psi}_{AB}\rangle|^2 = \boxed{\frac{1}{2}}$ (1)

ⓑ $|\tilde{\tilde{\psi}}_{AB}\rangle = |a_1\rangle\left(1|b_0\rangle + 0|b_1\rangle\right)$

$$|\langle b_0|\tilde{\tilde{\psi}}_{AB}\rangle|^2 = \boxed{1}\ (0); \qquad \langle b_1|\tilde{\tilde{\psi}}_{AB}\rangle = 0\ (1)\ !$$

Bob Measures First

$$|\psi_{AB}\rangle = \left(\frac{1}{2}|a_0\rangle + \frac{1}{\sqrt{2}}|a_1\rangle\right)|b_0\rangle + \left(\frac{1}{2}|a_0\rangle + 0|a_1\rangle\right)|b_1\rangle =$$

$\langle \text{Basis}|\text{Basis}\rangle =$

$$\left(\frac{1}{\sqrt{2}}|b_0\rangle + 0|b_1\rangle\right)^{+}\left(\frac{1}{\sqrt{2}}|b_0\rangle + 0|b_1\rangle\right) =$$

$$= \left(\frac{1}{\ }|a_0\rangle \ldots \sqrt{2}|a_1\rangle\right)\sqrt{3}|a\rangle + \left(1|a_0\rangle + 0|a_1\rangle\right)\frac{1}{\ }|b_1\rangle \qquad = \frac{1}{2}\langle b_0|b_0\rangle + 0\langle b_1|b_1\rangle$$

$\frac{1}{\sqrt{3}}$ ... $\frac{\sqrt{2}}{\sqrt{3}}|a_1\rangle)\frac{\sqrt{3}}{2}|b_0\rangle + (1|a_0\rangle + 0|a_1\rangle)\frac{1}{2}|b_1\rangle$

|basis⟩ Number |basis⟩ Number

$\frac{1}{\sqrt{2}}\langle b_0|b_1\rangle + 0\langle b_1|b_0\rangle = \frac{1}{2}$

0   0

$\sqrt{\ } - \frac{1}{\sqrt{\ }}$

← Bob measured	0 ($\frac{3}{4}$)		1 ($\frac{1}{2}$)	
← Alice	0	1	0	1
	$\frac{1}{3}$	$\frac{2}{3}$	1	0

Super-Luminal Communication

Can't entanglement be used for instantaneous communication

2. Spins in the above considered state

$$\frac{1}{2}|a_0\rangle|b_0\rangle + \frac{1}{2}|a_0\rangle|b_1\rangle + \frac{1}{\sqrt{2}}|a_1\rangle|b_0\rangle + 0|a_1\rangle|b_1\rangle$$

Suppose Alice measures the spin of her electron before Bob measures the spin

$$\frac{1}{\sqrt{2}}|a_0\rangle\left(\frac{1}{\sqrt{2}}|b_0\rangle + \frac{1}{\sqrt{2}}|b_1\rangle\right) + \frac{1}{\sqrt{2}}|a_1\rangle(1|b_0\rangle + 0|b_1\rangle)$$

She has $\frac{1}{2}, 0$, $\frac{1}{2}, 1$ probabilities

Suppose Now Bob makes the First Measurement

$$\left(\frac{1}{2}|a_0\rangle + \frac{1}{\sqrt{2}}|a_1\rangle\right)|b_0\rangle + \left(\frac{1}{2}|a_0\rangle + 0|a_1\rangle\right)|b_1\rangle$$

⇒ Make it unit vectors for Alice

$$\left(\frac{1}{\sqrt{3}}|a_0\rangle + \frac{\sqrt{2}}{\sqrt{3}}|a_1\rangle\right)\frac{\sqrt{3}}{2}|b_0\rangle + (1|a_0\rangle + 0|a_1\rangle)\frac{1}{2}|b_1\rangle$$

⇒ Bob has $\frac{3}{4}$ probability to measure 0, $\frac{1}{4}$ to measure 1

Alice has $\frac{1}{3}$ to measure 0   Alice has 1 measure 0

Total $\frac{3}{4}\cdot\frac{1}{3} + \frac{1}{4}\cdot 1 = \frac{1}{2}$

⇒ So even if Bob measures First Alice has same Probability to measure 0.!

⇒ Once measurement is done Entanglement is lost

... tell from her measurement

⇒ Alice can not [illegible]
whether they were made before or after Bob's

⇒ In some sense Difference Exist before
One makes the Measurement

— Say both made measurements

	P
00 →	$\frac{1}{4}$
01 →	$\frac{1}{4}$
10 →	$\frac{1}{2}$
11 →	0

Alice gets 0 P $\frac{1}{4}+\frac{1}{4}=\frac{1}{2}$

1 P $\frac{1}{2}+0=\frac{1}{2}$

Alice random string 0, 1
with equal $\frac{1}{2}$ probability

— This is same situation when she measured first before Bob measured

— So she cannot know if Bob measured or not

⇒ Same from the Bob's Point of view

⇒ So nobody will know who measured first

The Standard Basis for Tensor Product

— standard $\left(\overset{0}{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} \overset{1}{\begin{pmatrix} 0 \\ 1 \end{pmatrix}} \right)$ H $\left(\overset{0}{\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}} \overset{1}{\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}} \right)$

V

if both Alice and Bob use the standard basis

$$|\psi_{AB}\rangle = \overset{r}{\gamma_{00}} \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{R^2} \otimes \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{R^2} + \overset{s}{\gamma_{01}} \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{R^2} \otimes \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{R^2} + \overset{t}{\gamma_{10}} \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{R^2} \otimes \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{R^2} + \overset{u}{\gamma_{11}} \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{R^2} \otimes \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{R^2}$$

mapping in R^4

$\begin{bmatrix} 1 \\ \end{bmatrix}$ $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

R^4 - ordered basis

Rule of Construction

Rule

$$\underbrace{\begin{bmatrix} a_0 \\ a_1 \end{bmatrix} \otimes \begin{bmatrix} b_0 \\ b_1 \end{bmatrix}}_{R^2 \quad R^2} = \begin{bmatrix} a_0 \begin{bmatrix} b_0 \\ b_1 \end{bmatrix} \\ a_1 \begin{bmatrix} b_0 \\ b_1 \end{bmatrix} \end{bmatrix} = \underbrace{\begin{bmatrix} a_0 b_0 \\ a_0 b_1 \\ a_1 b_0 \\ a_1 b_1 \end{bmatrix}}_{R^4}$$

How to Entangle Qubits

1. Physically !

2. Mathematically: Using $\boxed{\text{CNOT}}$ Gate to Entangle Qubits

→ Standard ordered basis in $\boxed{R^4}$ space

$$\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right) \qquad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = I$$

- CNOT - gate comes from interchanging the order of the last two elements → results in the matrix for the CNOT gate

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

- This gate acts on pairs of qubits

Example. start by taking the unentangled tensor product

$$\underbrace{\frac{1}{\sqrt{2}} \underset{A}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} \underset{B}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} + \frac{1}{\sqrt{2}} \underset{A}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} \underset{B}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}}_{\text{unetangled}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} !$$

- Send this qubit through the CNOT Gate

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{\text{CNOT}} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \boxed{\text{entangled?}}$$

$$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \overset{A}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} \overset{B}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} + \frac{1}{\sqrt{2}} \overset{A}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} \overset{B}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}$$

$$= \left(\overbrace{\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 1 \end{bmatrix}}^{A} \right) \overset{B}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} + \left(\overbrace{0 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \end{bmatrix}}^{A} \right) \overset{B}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}$$

Bob did measurement

$$|\psi_{AB}\rangle = \frac{1}{\sqrt{2}} \underset{A}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} \underset{B}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} + \frac{1}{\sqrt{2}} \underset{A}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} \underset{B}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}$$

$$\begin{bmatrix} 1 & 0 \end{bmatrix} |\psi_{AB}\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$P \rightarrow 0 \quad \frac{1}{2}$

$P \rightarrow 1 \quad \frac{1}{2}$

$0 \quad 0 \quad [1] \quad [1]$

$\frac{1}{\sqrt{2}}\left(1\begin{bmatrix}1\\0\end{bmatrix}+0\begin{bmatrix}0\\1\end{bmatrix}\right)\begin{bmatrix}1\\0\end{bmatrix}$

$$\boxed{|\tilde{\psi}_{AB}\rangle = \frac{1}{\sqrt{2}}\underset{A}{\begin{bmatrix}1\\0\end{bmatrix}}\underset{B}{\begin{bmatrix}1\\0\end{bmatrix}} + \frac{1}{\sqrt{2}}\underset{A}{\begin{bmatrix}0\\1\end{bmatrix}}\underset{B}{\begin{bmatrix}0\\1\end{bmatrix}}} \Leftarrow$$

Show that this is an entangled state.

(a) Bob is not in the picture

$$P = 0 ? \quad \left|[1\,0]\,|\tilde{\psi}_{AB}\rangle\right|^2 \overset{?}{=} \frac{1}{\sqrt{2}}\overset{A}{[1\,0]}\overset{A}{\begin{bmatrix}1\\0\end{bmatrix}}\begin{bmatrix}1\\0\end{bmatrix} + \frac{1}{\sqrt{2}}\cancel{\overset{A}{[1\,0]}\overset{A}{\begin{bmatrix}0\\1\end{bmatrix}}}^{\,0}\overset{B}{\begin{bmatrix}0\\1\end{bmatrix}}$$

$$= \left|\frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\end{bmatrix}\right|^2 = \ = \frac{1}{2}$$

$$P = 1 \quad \left|[0\,1]\,|\tilde{\psi}_{AB}\rangle\right| = \left|\frac{1}{\sqrt{2}}\cancel{\overset{A}{[0\,1]}\overset{A}{\begin{bmatrix}1\\0\end{bmatrix}}}_{0}\overset{B}{\begin{bmatrix}1\\0\end{bmatrix}} + \frac{1}{\sqrt{2}}\cancel{\overset{A}{[0\,1]}\overset{A}{\begin{bmatrix}0\\1\end{bmatrix}}}_{1}\overset{B}{\begin{bmatrix}0\\1\end{bmatrix}}\right|^2$$

$$= \frac{1}{2}\left|\begin{bmatrix}0\\1\end{bmatrix}\right|^2 = \frac{1}{2}$$

(b) Bob did the measurement, 0 - 6 ibery

$$|\tilde{\psi}_{AB}\rangle = \begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}$$

$P = 0 \qquad ? \qquad 100\%$

$P = 1 \qquad 0\%$