Spin_orbit

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— Electron is in magnetic field of moving nucleus with charge of Ze.

Let's use classical physics

Moving point charge creates a magnetic field according to Biot-Savart Law

$$B = \frac{\mu_0}{4\pi} q \cdot \frac{\vec{v} \times \vec{r}}{r^3}$$

— We apply above formula to atom considering its effect on electron

(We also use our notation according to which Coulomb Force is $F = k\frac{q_1 q_2}{r^2} = \frac{q_1 q_2}{4\pi\varepsilon_0 r^2} \xrightarrow{\text{New def}} \frac{q_1 q_2}{r^2}$)

$\frac{q}{\sqrt{4\pi\varepsilon_0}} \rightarrow q$

$$B_N = \frac{\mu_0}{4\pi} Ze \frac{\vec{V}_N \times \vec{r}}{r^3}$$

remember $\sqrt{\varepsilon_0 \mu_0} = \frac{1}{c} \Rightarrow \mu_0 = \frac{1}{c^2 \varepsilon_0}$

$$B_N = \frac{Ze \vec{V}_N \times \vec{r}}{c^2 4\pi\varepsilon_0 r^3} \xrightarrow{\text{with our definition}} \frac{Ze \vec{V}_N \times \vec{r}}{c^2 r^3} = \frac{\vec{V}_N \times \vec{E}_N}{c^2}$$

— Now we use the fact that $\vec{V}_N = -\vec{V}_e$

$$B_N = -\frac{\vec{V}_e \times \vec{E}}{c^2} = -\frac{Ze \vec{V}_e \times \vec{r}}{c^2 r^3} =$$

$$B_N = -\frac{Ze m \vec{V}_e \times \vec{r}}{mc^2 r^3} = \frac{-Ze \overbrace{\vec{P}_e \times \vec{r}}^{-\vec{L}}}{mc^2 r^3} = \frac{Ze \vec{L}}{mc^2 r^3}$$

— We learned that if electron spins around Nucleus it has magnetic momentum $\mu_L = -\frac{e}{2m_e} \cdot \vec{L}$

— In the rest frame of electron $\mu_L = 0$

However electron has internal angular momentum = Spin

Thus One Expects $\mu_s = -g\frac{e}{2m_e} \vec{S}$ -

— g - gyroscopic Factor

— μ_s - will induce interaction

$$\underbrace{H_{LS}}_{\text{Larmor}} = -\mu_s B_N = \frac{gZe^2}{2m^2c^2} \frac{\vec{S} \cdot \vec{L}}{r^3}$$

— g - gyroscopic Factor — gyromagnetic Ratio

— Thomas Precession

Orbits of electron is curvilinear. To transform to electron rest Frame — Lorentz transformation and Rotation

$w_T = \frac{1}{c^2}\left(\frac{\gamma^2}{\gamma+1}\right) a \times v$ $\quad \Omega = -\omega(\gamma-1)$

NR Limit

$w_T \approx \frac{1}{2c^2} a \times v \qquad \vec{a} = \frac{d\vec{v}}{dt}$

$$H_T = -\frac{Ze^2}{2m^2c^2}\frac{\vec{S}\times\vec{L}}{r^3} = \Omega_T \cdot S$$

— $H_{LS} = H_{Larmor} + H_T =$

$$H_{LS} = \frac{(g-1)\,Ze^2}{2m^2c^2}\frac{\vec{S}\cdot\vec{L}}{r^3}$$

— $g = 2$

$$H_{LS} = \frac{Ze^2}{2m^2c^2}\frac{\vec{S}\cdot\vec{L}}{r^3}, \qquad \frac{e^2}{\hbar c} = \alpha$$

$$H_{LS} = \frac{Z\alpha\hbar c}{2m^2c^2}\frac{\vec{S}\cdot\vec{L}}{r^3} = \frac{Z\alpha\hbar}{2m^2c}\frac{\vec{S}\cdot\vec{L}}{r^3}$$

— Bohr magneton $\mu_B = \frac{e\hbar}{2mc}$

$\Rightarrow$ Real Hamiltonian in the Hydrogen Like atom

$$H = \frac{p^2}{2m} - \frac{Ze^2}{r} + \frac{Ze^2}{2m^2c^2}\frac{\vec{S}\cdot\vec{L}}{r^3} =$$

$$= \frac{p^2}{} - Z\alpha\hbar c + \frac{Z\alpha\hbar}{}\ \vec{S}\cdot\vec{L}$$

$$2m \qquad r \qquad 2m^2c \qquad r^3$$

→ We treat $H_0 = \frac{p^2}{2m} - \frac{Ze^2}{r}$

$$E_n^0 = -\frac{Z^2\alpha^2 mc^2}{2n^2}$$

→ First order Perturbation Theory

$$H' = \frac{Z\alpha\hbar}{2m^2c}\frac{\vec{S}\cdot\vec{L}}{r^3} = H_{LS}$$

$$\Delta_n^{(1)} = \langle \phi_{Jm}^{n\ell} | \bar{H}_{LS} | \phi_{Jm}^{n\ell} \rangle =$$

$$\frac{\alpha^3 m^3 c^3}{\hbar^3} = \frac{1}{a^3}$$

$$= \frac{Z\alpha\hbar}{2m^2c} \langle \phi_{Jm}^{n\ell} | \frac{\vec{S}\vec{L}}{r^3} | \phi_{Jm}^{n\ell} \rangle$$

$[J, \vec{S}\cdot\vec{L}]$

$$\Delta_n^{(1)} = \frac{Z\alpha\hbar}{2m^2c} \langle \phi_{Jm}^{n\ell} | \frac{1}{r^3} | \phi_{Jm}^{n\ell} \rangle \underbrace{\langle \phi_{Jm}^{n\ell} | \vec{S}\vec{L} | \phi_{Jm}^{n\ell} \rangle}_{II}$$

$$a_0 = \frac{\hbar^2}{e^2 m_e} = \frac{\hbar^2}{\alpha\hbar c m} = \frac{\hbar}{\alpha c m}$$

$$\frac{e^2}{\hbar c} = \alpha$$

$$\langle \phi_{Jm}^{n\ell} | \frac{1}{r^3} | \phi_{Jm}^{n\ell} \rangle = \frac{Z^3\alpha^3 m^3 c^3}{\hbar^3 n^3 \ell(\ell+1)(\ell+\frac{1}{2})} \qquad I = \frac{\alpha^3 m^3}{n^3 \ell(\ell+1)(\ell+\frac{1}{2})}$$

$$a_B = \frac{\hbar^2}{e^2 m_e} = \frac{\hbar}{\alpha m c}$$

$$\langle \phi^{n\ell} | \frac{1}{r} | \phi^{n\ell} \rangle = \frac{Z^3}{\ldots} \frac{1}{\ldots}$$

$-\langle \phi_{jm} | \frac{1}{r^3} | \phi_{jm} \rangle \quad a_B^3 n^3 \, \ell(\ell+1)(\ell+\frac{1}{2})$

— Natural Units. $\hbar = 1 \quad c = 1$

$$- \langle \phi_{jm}^{n\ell} | \vec{S}\vec{L} | \phi_{jm}^{n\ell} \rangle = \frac{1}{2} \langle \phi_{jm}^{n\ell} | J^2 - L^2 - S^2 | \phi_{jm}^{n\ell} \rangle = //$$

$$\vec{J} = \vec{L} + \vec{S} \Rightarrow J^2 = L^2 + S^2 + 2\vec{L}\vec{S}$$

$$\vec{L}\vec{S} = \frac{J^2 - L^2 - S^2}{2}$$

$$// = \frac{1}{2}\left(j(j+1) - \ell(\ell+1) - \frac{3}{4} \right) = \langle \phi | \vec{S}\vec{L} | \phi \rangle$$

$\boxed{\text{if } j = \ell + \frac{1}{2}}$

$$\ell^2 + \frac{3}{2}\ell + \frac{1}{2}\ell + \frac{3}{4}$$

$$\langle \phi | \vec{S}\vec{L} | \phi \rangle^{j=\ell+\frac{1}{2}} = \frac{1}{2}\left((\ell+\tfrac{1}{2})(\ell+\tfrac{3}{2}) - \ell^2 - \ell - \frac{3}{4} \right) = \frac{\ell}{2}$$

$\boxed{\text{if } j = \ell - \frac{1}{2}}$

$$\ell^2 - \frac{1}{4}$$

$$\langle \phi | \vec{S}\vec{L} | \phi \rangle^{j=\ell-\frac{1}{2}} = \frac{1}{2}\left((\ell-\tfrac{1}{2})(\ell+\tfrac{1}{2}) - \ell^2 - \ell - \frac{3}{4} \right) = \frac{-\ell-1}{2}$$

— Combining I and II

$$\langle \ldots \rangle^{j=\ell+\frac{1}{2}}$$

$$\Delta_n^{(1)} = \frac{Z^4 \alpha \hbar}{2m^2c} \frac{\alpha^3 m^3 c^3}{\hbar^2 n^3 \ell(\ell+1)(\ell+\frac{1}{2})} \frac{1}{2} \begin{pmatrix} \ell \\ -\ell-1 \end{pmatrix} \begin{array}{l} j = \ell + \frac{1}{2} \\ j = \ell - \frac{1}{2} \end{array}$$

$$\Delta_n^{(1)} = \frac{Z^4 \alpha^4 m}{4 n^3 \ell(\ell+1)(\ell+\frac{1}{2})} \begin{pmatrix} \ell \\ -(\ell+1) \end{pmatrix} \begin{array}{l} j = \ell + \frac{1}{2} \\ j = \ell - \frac{1}{2} \end{array}$$

Relativistic Correction:

$$T = E_{TOT} - mc^2 = \sqrt{m^2c^4 + p^2c^2} - mc^2$$

$$T = mc^2 \sqrt{1 + \frac{p^2}{m^2c^2}} - mc^2$$

$$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{1}{8}x^2$$

$$f(0) + f'(0)x + f''(0)\frac{x^2}{2}$$

$$f'(x) = \frac{1}{2}\frac{1}{\sqrt{1+x}}$$

$$f''(x) = -\frac{1}{4}\frac{1}{(1+x)^{3/2}}$$

$$T = mc^2\left(1 + \frac{p^2}{2m^2c^2} - \frac{1}{8}\frac{p^4}{m^4c^4}\right) - mc^2$$

$$= \frac{p^2}{2m} - \frac{1}{8}\frac{p^4}{m^3c^2}$$

$$H_{kin} = -\frac{1}{2m}\frac{p^4}{4m^2c^2} = -\frac{1}{2mc^2}T_0^2 =$$

$$= -\frac{1}{2mc^2}(H_0 - V)^2 =$$

$$= -\frac{1}{2mc^2}\left(H_0^2 - H_0V - VH_0 + V^2\right)$$

$2mc^2$

$$\langle \phi_{Jm} | H_0^2 - V H_0 - H_0 V | \phi_{Jm} \rangle = -3(E_n^0)^3$$

$$\langle \phi_{Jm} | V | \phi_{Jm} \rangle = 2E_n^0$$