Lecture 2: Classical Space-Time Symmetris

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1 Introduction

The reality exists as we observe it and our observation reveals some features which in our perception has to do with aesthetics and harmony. Systems look to us *beautiful* due to their geometrical proportions or sphericity: some systems perceived the same independent from the point of our observation, some systems look the same if one moves back or fort with in the space and time and some systems show periodic structure again in the space and the time (like waves). We find some time that inverting space and time keeps the observation the same or even more subtile, such as "happenings" in the system does not depend on references frames we choose, those which are moving with constant velocity with respect to each other.

All these in which some transformations do not change some observations we associate with specific symmetries being observed in physical systems. On the other hand the fact of some observations not being changed due to some transformations can be associate with certain quantities which are conserved due to considered transformations.

With this chain of logics we can make our first formulation of

the relations between symmetry and conservation of certain "observables" with respect to the certain "transformation".

With above formulation, the discussion of the symmetry of reality requires a consideration of *three* main parts: (i) Physical system, (ii) reference frame with respect of which this system is observed, and (iii) modification of the "observable" due to transformation of the system with respect to the reference frame.

The transformations which we are interested in are related to the space-time and can be categorized as follows:

- Rotation in Space
- Space Translation
- Time Translation
- Space Inversion

- Time Inversion
- Change of Inertial Reference Frame

Building Blocks of Reality: Before to start the mathematical formulation of the concept of symmetry, we first postulate that in classical mechanics, the *building blocks* of reality are *positions,* r, *velocities* $\vec{v}$ *(or alternatively momenta* $\vec{p}$) and *time,* t at which they are measured.

Paradigm of Transformations: As it follows from the above discussions the concept of *symmetry* is closely related to the change of our "viewpoint" of the considered systems which can be achieved by appropriate transformation of the observed system. However the same change of the "viewpoint" can be achieved also by appropriate change (or transformation) of the reference frame were the observation is being made. We further elaborate *two* kind of transformations: *active* transformation in which the observed system is transformed in the space and time at given reference frame and *passive* transformation, in which the reference frame is transformed in the *"opposite direction"* of *active* transformation.

Now the Paradigm of transformations is the equivalence between *Active* and *Passive* transformations.

2 Space-Time Transformations

We will try not to develop a mathematical framework for description of space-time transformations which is most relevant for studying classical mechanical systems.

We consider the coordinate $r_i(t)$, momentum $p_i(t)$ and time t as the most fundamental level of information that we can gather about the system under the observation. As such the measured "observable" of the system A will be function of these set of variables: $A(r_i(t), p_i(t), t)$. Note that for simplicity we are considering a scalar observable A that is characterized only by its magnitude.

The space-time transformation that we are interested ar:

Space Translations:

$$\begin{aligned} &\vec{r} \to \vec{r} + \vec{a}; \quad \vec{p} \to \vec{p}; \quad t \to t \\ &A(\vec{r}, \vec{p}, t) \to A(\vec{r} + \vec{a}, \vec{p}, t), \end{aligned} \tag{1}$$

Time Translations:

$$\begin{aligned} &\vec{r}(t) \to \vec{r}(t+\epsilon); \quad \vec{p}(t) \to \vec{p}(t+\epsilon); \quad t \to t+\epsilon \\ &A(\vec{r}(t), \vec{p}(r), t) \to A(\vec{r}(t+\epsilon), \vec{p}(t+\epsilon), t+\epsilon), \end{aligned} \tag{2}$$

Rotations: Rotation by an angle θ about the direction of a unit vector $\hat{n}$. This results to the change of the measured coordinates and momenta in a specific form

$$\begin{aligned} &r_i \to \tilde{r}_i'; \quad p_i \to \tilde{p}_i'; \quad t \to t \\ &A\big(\vec{r}(t), \vec{p}(t), t\big) \to A(\vec{r}'(t), \vec{p}'(t), t), \end{aligned} \tag{3}$$

Galilean Boosts: Galilean Boosts change the observed coordinates and momenta in the system according to Galilean transformation:

$$\begin{aligned} &\vec{r} \to \vec{r} + \vec{V}t; \quad \vec{p} \to \vec{p} + m\vec{V}; \quad t \to t \\ &A\big(\vec{r}(t), \vec{p}(t), t\big) \to A\big(\vec{r} + \vec{V}t, \vec{p} + m\vec{V}, t\big) \end{aligned} \tag{4}$$

Space Reflection: In this case one reverses all the measured coordinates and momenta:

$$\begin{aligned} &\vec{r} \to -\vec{r} \quad \vec{p} \to -\vec{p} + m\vec{V}; \quad t \to t \\ &A\big(\vec{r}(t), \vec{p}(t), t\big) \to A\big(- \vec{r}, -\vec{p}, t\big) \end{aligned} \tag{5}$$

Time Inversion: In this case one reverses the measured time int the system:

$$\begin{aligned} &\vec{r}(t) \to \vec{r}(-t) \quad \vec{p}(t) \to \vec{p}(-t); \quad t \to -t \\ &A\big(\vec{r}(t), \vec{p}(t), t\big) \to A\big(\vec{r}(-t), \vec{p}(-t), -t\big) \end{aligned} \tag{6}$$

The one of the cornerstones of the foundation of Classical Mechanics is that Fundamental Physical Laws does not change due to above transformations and apparent changes should have an emergent nature.

Digress: Internal Symmetries. While Classical Mechanics allows symmetries only in a space and time, the quantum mechanics reveals a new reality for symmetries. Those symmetries does not have analogy with our classical perception of symmetry. These are *Charge Conjugation, matter-antimatter symmetry, isotope symmetry, gauge symmetry etc.*.

3 Operators of Transformations

To define the above discussed transformations mathematically one introduces *operators* that act upon the observed system (active transformation) or the reference frame (passive transformation) through the "building blocks" of reality - coordinates, momenta and time.

These operators we can formulate by their action on coordinates as fol

Transformation	Operator	Action on Coordinates	Action on Momenta
Space Translation	$T(a)$	$r_i \to r_i + a$	$p_i \to p_i + a$
Time Translation	$U(t_0)$	$r_i(t) \to r_i(t+t_0)$	$p_i(t) \to p_i(t+t_0)$
Rotation	$R(\hat{n},\theta)$	$r'_m = \sum_{n=1}^{3} R_{mn}(\hat{n},\theta) r_n$	$p'_m = \sum_{n=1}^{3} R_{mn}(\hat{n},\theta) p_n$
Galilean Boost	$G(v)$	$r_i \to r_i + v \cdot t$	$p_i \to p_i + m \cdot v$
Space Inversion	$\mathcal{R}$	$r_i \to -r_i$	$p_i \to -p_i$
Time Inversion	$\mathcal{T}$	$r_i(t) \to r_i(-t)$	$p_i(t) \to -p_i(-t)$

3.1 Observables

In the above definitions: operators act on the coordinates, r_i and momenta p_i which are the observables of classical mechanics. Or more precisely, any observables of the system is described through the coordinates and momenta (velocities). This statement is one of the foundation of the classical mechanics for which the degrees of freedom is defined by $r_i(t)$ and $p_i(t)$ $(v_i(t))$. This assumes that one can not go deeper or no more fundamental layer of reality exists beyond the coordinates and momenta.

3.2 Properties of Operators

Our next task is mathematical formulation of the properties of operators. For this we observe that all above discussed transformations satisfy the definition of the groups with its own rules of composition. Thus our further discussion will be based on some properties of group theory. Before go further we notice that the above discussed transformation can be grouped into continuous and discrete transformation, were continuous are space, time translation, rotation, Galilean boost, while discrete are space reflection and time inversion. We will start first discussing operators of continuos transformations.

3.3 Space Translation

$$T(a_1)T(a_2) = T(a_1 + a_2) = T(a_2 + a_1) = T(a_2)T(a_1) \tag{7}$$

for acting on both r_i and p_i.

Identity: $T(0) = 1$

Continuos Group: The properties of the group are defined by the properties of the operator at the neighborhood of identity. Or in human words: any finite transformation is defined through the small/infinitesimal transformations, i.e.

$$T(a) = T(\epsilon)^{\frac{a}{\epsilon}} \tag{8}$$

where ϵ is very small. As a result we can expand:

$$T(\epsilon) = 1 + T'(0)\epsilon + \frac{1}{2}t'')(0)\epsilon^2 + \cdots . \tag{9}$$

In the limit of $\epsilon \to 0$ one needs only to know $T'(0)$.

Example: Let A be an observable. Consider a small transformation in one direction: $T(\epsilon)$. In this case $T(\epsilon) : r_i = r'_i = r_i + \epsilon$ $(\delta r_i = r'\_r_{=}\epsilon$ and $T(\epsilon) : p_i = p_i$.

Now $\boxed{T(\epsilon)A = A' = A(r', p')}$. Then

$$\delta A = A' - A = \sum_{n,k} \left(\frac{\partial A}{\partial r_{n,k}} \delta r_{n,k} + \frac{\partial A}{\partial p_{n,k}} \delta p_{n,k} \right) = \sum_{n,k} \frac{\partial A}{\partial r_{n,k}} \epsilon_k = \vec{\epsilon} \cdot \sum_n \nabla_n A, \tag{10}$$

where n counts the number of constituents in the system and k the components in the reference frame. Thus we obtained that operator of space translation acted on the observable A is described through ∇ operators which we will call *generators*.

3.4 Rotations

Some properties:

- Rotations do not commute
- **Euler Theorem:** Any Rotation leaves some axis unchanged - therefore any rotation can be described by $\hat{n}$ and θ where
$$\hat{R}\hat{n} = \hat{n} \tag{11}$$

Example: Rotation about $\hat{z}$ axis.

$$\begin{aligned} x' &= x \cdot cos(\theta) - y \cdot sin(\theta) \\ y' &= x \cdot sin(\theta) + y \cdot cos(\theta) \\ z' &= 0 \end{aligned} \tag{12}$$

In the matrix form it can be written as:

$$r' = \hat{R}(\hat{n}_z, \theta) \cdot r \tag{13}$$

where $r = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $r' = \begin{pmatrix} x' \\ y\prime \\ z' \end{pmatrix}$ and

$$\hat{R}(\hat{n}_z, \theta) = \begin{pmatrix} cos(\theta) & -sin(\theta) & 0 \\ sin(\theta) & cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{14}$$

for small rotation angle $\theta \to \epsilon$ one obtains

$$\hat{R}(\hat{n}_z, \epsilon) = I + \begin{pmatrix} 0 & -\epsilon & 0 \\ \epsilon & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{15}$$

Using Eq.(15) in Eq.(13) one obtains:

$$\begin{aligned} x' &= x - \epsilon y \\ y' &= y + \epsilon x \\ z' &= z \end{aligned} \tag{16}$$

If we now represent the above transformation in the form of $\vec{r'} = \vec{r} + \vec{\delta r}$ then for $\vec{\delta r}$ one obtains:

$$\vec{\delta r} = \epsilon \hat{n}_z \times \vec{r} \tag{17}$$

which now can be generalized for rotation around any axis $\hat{n}$ as:

$$r' = \vec{r} + \epsilon \hat{n} \times \vec{r} \quad \text{and} \quad \delta r_i = \epsilon \sum_{j,k} \epsilon_{ijk} n_j r_k, \tag{18}$$

where ϵ_{ijk} represents the asymmetric Lev-Civita matrices of the rank 3.

The above equation is true for momentum with

$$\delta p_i = \epsilon \sum_{j,k} \epsilon_{ijk} n_j p_k. \tag{19}$$

Vectors: In fact this relation can be used as a definition of the vector. That is, V is a vector if due to infinitesimal rotation ϵ around any axis $\hat{n}$ it changes according

$$V' = \vec{V} + \epsilon \hat{n} \times \vec{V} \quad \text{and} \quad \delta V_i = \epsilon \sum_{j,k} \epsilon_{ijk} n_j V_k, \tag{20}$$

Example: Show that $\vec{L} = \vec{r} \times p$ is a vector.

Scalar: If the observable is invariant under the operation of rotation:

$$S' = \hat{R} S = S \tag{21}$$

3.4.1 Fields:

The question we address now is how the fields change under operation of rotation. The fields can be as scalar, vector, tensor etc, defined according to their response with respect to the rotation, R.

Scalar Fields: The field (say $\phi(r,t)$) as scalar if it satisfies the following relation:

$$\hat{R}\cdot\phi(r,t)=\phi(r^{"},t) \tag{22}$$

where

$$r_i^{"}=\sum_j R_{i,j}^{-1}r_j \tag{23}$$

Vector Fields: The field (say $\vec{A}(r,t)$) as scalar if it satisfies the following relation:

$$A_i'(r',t)=\sum_j R_{ij}A_j(r,t) \tag{24}$$

where

$$r_i'=\sum_j R_{ij}r_j \tag{25}$$

3.5 Reflection and Rotation

We can differentiate between real and pseudo- scalars and vectors. The scalarity is defined with respect to the rotation while *real* scalar is the one which is even with respect to the space reflection. Accordingly *pseudo*-scalar is the quantity which is odd (changes the sign) with respect to the space reflection.

Similarly, one define *real*- and *pseudo*- vectors. In which again vector-ness is defined with respect to the rotation and real vector is odd and pseudo vector is even with respect to the space reflection.

Example: Electric vector potential is a real vector:

$$\vec{A}(r,t)\rightarrow-\vec{A}(-r,t) \tag{26}$$

while the Magnetic field is a pseudo-vector

$$\vec{B}(r,t)=\rightarrow\vec{B}(-r,t) \tag{27}$$

3.6 Rotation Matrices

Rotation matrices are in general not commutative:

$$\hat{R}(\hat{n}_1,\theta_1)\hat{R}(\hat{n}_2,\theta_2)\neq\hat{R}(\hat{n}_2,\theta_2)\hat{R}(\hat{n}_1,\theta_1) \tag{28}$$

but

$$\hat{R}(\hat{n},\theta_1)\hat{R}(\hat{n},\theta_2) = \hat{R}(\hat{n},\theta_2)\hat{R}(\hat{n},\theta_1) \tag{29}$$

Using the latter property we always can present

$$\hat{R}(\hat{n},\theta) = \prod_{i=1}^{N} \hat{R}(\hat{n},\epsilon) \tag{30}$$

where $\epsilon = \frac{\theta}{N}$. If now we take $N \to \infty$ and use the fact that

$$\hat{R}(\hat{n},0) = I \tag{31}$$

we notice that all the properties of rotation can be identified at the neighborhood of unity matrix I.

This statement is more obvious if one introduce matrix $J(\hat{n})$ which is called *Generator* of rotation around the axis $\hat{n}$ such that

$$R(\hat{n},\epsilon) = e^{-iJ(\hat{n})\epsilon}, \tag{32}$$

Then according to Eq.(30) any rotation matrix is defined if its generator is defined:

$$R(\hat{n},\theta) = e^{-iJ(\hat{n})\theta}. \tag{33}$$

Note that the matrix exponent is understood as $e^M = I + M + \frac{1}{2}M + \cdots \frac{1}{n!}M^n$

3.7 Generators of Rotation

We now discuss several properties of the generators of rotation J.

We first notice that since rotation matrices R are real the generators J are imaginary orators. Then using the property of R that

$$R^T = R^{-1} \tag{34}$$

we obtain:

$$(e^{-iJ})^T \equiv e^{-iJ^T} = e^{iJ} \tag{35}$$

therefore $J^T = -J$. Using the fact that J is imaginary we obtain that J's are Hermitean operators:

$$J^\dagger = J \tag{36}$$

Using the earlier example of infinitesimal rotation around $\hat{z}$ in Eq.(15) one notice that J is an antisymmetric matrix with diagonal elements being zero. From this we conclude that only three parameters are needed to completely describe rotational matrices. From this it follows that rotation can be expressed as linear combination of 3 independence generators: J's. We define these three independent generators the once representing rotation around $\hat{x}$, $\hat{y}$ and $\hat{z}$ axises (named by J_x, J_y and J_z respectively.

Let us now obtain the expressions for J_i's:

We start with discussing the rotation of the vector r_i such that:

$$r'_i = \sum_j R_{ij} r_j = \sum_j \left(e^{-i\epsilon J(\hat{n})} \right)_{ij} r_j = r_i - \sum_j i\epsilon J(\hat{n})_{ij} r_j, \tag{37}$$

where in the last step we used the expansion $e^{-i\epsilon J(\hat{n})} \approx I - i\epsilon J(\hat{n})$. From the above equation we obtain:

$$\delta r_i = -\epsilon \sum_j i\epsilon J(\hat{n})_{ij} r_j. \tag{38}$$

Using this equation and the definition of δr from Eq.(18) one obtains:

$$\sum_{j,k} \epsilon_{ijk} n_j r_k = -\sum_k i\epsilon J(\hat{n})_{ik} r_k \tag{39}$$

where in the last step we renamed $j \to k$ in Eq.(38).

Let us now fix the axis of rotation in some given direction j: $\hat{n} \to n_j$ and then define :

$$J_j n_j \equiv J(n_j)_{ik}. \tag{40}$$

Then from Eq.(39) one obtains:

$$\epsilon_{ijk} n_j r_k = -iJ(n_j)_{ik} r_k = -i(J_j)_{ik} n_j r_k \tag{41}$$

The above equation we can solve for $(J_j)_{ik}$:

$$(J_j)_{ik} = i\epsilon_{ijk} = -i\epsilon_{jik}. \tag{42}$$

Redefining $j \leftrightarrow i$ one obtains the final expression for the generators of the rotation:

$$(J_i)_{jk} = -\epsilon_{ijk}. \tag{43}$$

For general direction of rotation $\hat{n}$:

$$J(\hat{n}) = \sum_i n_i J_i. \tag{44}$$

Using the expression of $(J_i)_{jk}$ from Eq.(43) one can present the generators in the matrix form as follows:

$$J_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad J_y = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} \quad J_z = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{45}$$

Using the explicit form of the generators it is easy to show that they satisfy the following comutating relations:

$$[J_i J_j] = i \sum_k \epsilon_{ijk} J_k, \tag{46}$$

which is also called the *algebra* of generators of rotation.

4 Home Works

4.1 Angular Momentum is a Pseudo-Vector

First we show that it is a vector, applying the vector definition in terms of the operation of the rotation: according to which A_i is a vector if it changes during the rotation around the axis $\hat{n}$ according to the relation:

$$\delta A_i = \epsilon \sum_{j,k} \epsilon_{ijk} n_j A_k \tag{47}$$

.

Now defining the angular momentum as:

$$L_i \equiv [r \times p]_i = \sum_{j,k} \epsilon_{ijk} r_j p_k \tag{48}$$

where r_j and p_k are the components of the vectors of $\vec{r}$ and $\vec{p}$, we need to show that after the rotation

$$L_i' = R_{i,j}(\hat{n}, \epsilon) L_j = L_i + \delta L_i \tag{49}$$

where

$$\delta L_i = \epsilon \sum_{j,k} \epsilon_{ijk} n_j L_k. \tag{50}$$

To prove this relation we use Eq.(48) to express L' as

$$L_i' = \sum_{j,k} \epsilon_{ijk} r_j' p_k' = \sum_{j,k} \epsilon_{ijk} (r_j + \delta r_j)(p_k + \delta p_k). \tag{51}$$

Now using the fact that r_i and p_i are vectors and they change during the rotation according to Eq.(47), for infinitesimal rotation angle ϵ (neglecting ϵ^2 term) one obtain:

$$\delta L_i = \sum_{j,k} \epsilon_{ijk} r_j \delta p_k + \sum_{j,k} \epsilon_{ijk} \delta r_j p_k = \epsilon \sum_{j,k} \sum_{m,n} \epsilon_{ijk} \epsilon_{kmn} n_m r_j p_n + \epsilon \sum_{j,k} \sum_{m,n} \epsilon_{ijk} \epsilon_{jmn} n_m r_n p_k. \tag{52}$$

Now using the antisymmetric properties of Levi-Civita matrices the above equation we can present in the following form

$$\delta L_i = \epsilon \sum_{j,m,n} \sum_{k} \epsilon_{kij} \epsilon_{kmn} n_m r_j p_n - \epsilon \sum_{k,m,n} \sum_{j} \epsilon_{jik} \epsilon_{jmn} n_m r_n p_k \tag{53}$$

then use the relation:

$$\sum_i \epsilon_{ijk} \epsilon_{imn} = \delta_{jm} \delta_{kn} - \delta_{jn} \delta_{km} \tag{54}$$

to write for Eq.(53)

$$\delta L_i = \epsilon \sum_{j,m,n} (\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}) n_m r_j p_n - \epsilon \sum_{k,m,n} (\delta_{im}\delta_{kn} - \delta_{in}\delta_{km}) n_m r_n p_k = \quad (55)$$

$$\epsilon \sum_{j,m,n} (\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}) n_m r_j p_n - \epsilon \sum_{j,m,n} (\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}) n_m r_n p_j = \quad (56)$$

$$\epsilon \sum_{j,m,n} (\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}) n_m r_j p_n - \epsilon \sum_{j,m,n} (\delta_{im}\delta_{jn} - \delta_{ij}\delta_{nm}) n_m r_j p_n = \quad (57)$$

$$\epsilon \sum_{j,m,n} (\delta_{ij}\delta_{mn} - \delta_{in}\delta_{mj}) n_m r_j p_n = \epsilon \sum_{j,m,n} \sum_{k} \epsilon_{kim}\epsilon_{kjn} n_m r_j p_n = \quad (58)$$

$$\epsilon \sum_{m} \sum_{k} \epsilon_{kim} n_m L_k = \epsilon \sum_{m,k} \epsilon_{imk} n_m L_k, \quad (59)$$

which proves the relation of Eq.(50). In derivation of above relations we did the following procedures: in the second part of (56) we renamed the index $k \to j$, then in (57) in the second part we replaced $n \leftrightarrow j$, this resulted to the first part of the (58) where we used the relation of Eq.(54) to obtain the second part. And finally we used the definition of the angular momentum of Eq.(48) and permutation property of Levi-Civita matrices in (59) part to arrive the final expression.

So far we proved that the angular momentum has a vector properties with respect to the rotation. To prove that it is a pseudo-vector we consider the operator of space inversion and use the relation of Eq.(48). This results to

$$\mathcal{R}\vec{L} = \mathcal{R}(\vec{r} \times \vec{p}) = (-\vec{r} \times (-\vec{p})) = \vec{L} \quad (60)$$

where we used the fact that $\vec{r}$ and $\vec{p}$ are true vectors as it is defined in definition of the space reflection in Sec. (3)