Hermann Minkowski and Relativity as a Rotation

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Overview

- Transition from Newtonian to Einsteinian kinematics
- Minkowski spacetime as a geometric unification
- Lorentz transformations as hyperbolic rotations
- Rapidity, invariance, and metric structure
- Proper time and geodesics
- Light cones, causal structure
- 4-vectors and invariants in relativistic dynamics
- Minkowski diagrams and spacetime geometry

Newtonian vs. Einsteinian Spacetime

Galilean transformations:

$$t' = t, \qquad x' = x - vt$$

imply:

$$\Delta t' = \Delta t$$

- Time is absolute; simultaneity is universal.

But Maxwell's equations enforce:

$$c' = c$$

inconsistent with Galilean relativity.

Einstein's Two Postulates

1. Laws of physics are identical in all inertial frames.
2. Speed of light is invariant: $c =$ constant.

Leads to Lorentz transformations:

$$ct' = \gamma(ct - \beta x), \qquad x' = \gamma(x - \beta ct)$$

$$\gamma = \frac{1}{\sqrt{1 - \beta^2}}, \quad \beta = \frac{v}{c}$$

Minkowski Spacetime

- Events represented by 4-vectors: $(x^\mu) = (ct, x, y, z)$
- Spacetime interval:

$$s^2 = (ct)^2 - x^2 - y^2 - z^2$$

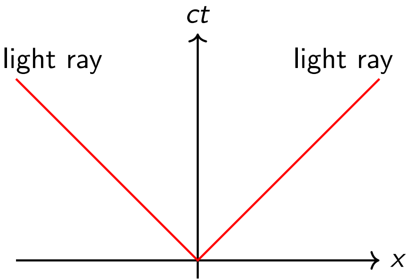
- Minkowski metric:

$$\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$$

- Lorentz transformations preserve s^2

$$s'^2 = \eta_{\mu\nu} x'^\mu x'^\nu = \eta_{\mu\nu} x^\mu x^\nu = s^2$$

Spacetime Diagram (Light Cone)



- Light cone divides spacetime into causal regions.
- All physical motion lies inside or on the cone.

Lorentz Transformations as Hyperbolic Rotations

Boost in terms of rapidity ϕ:

$$\tanh\phi = \beta, \qquad \gamma = \cosh\phi, \qquad \beta\gamma = \sinh\phi$$

Then:

$$\begin{pmatrix} ct' \\ x' \end{pmatrix} = \begin{pmatrix} \cosh\phi & -\sinh\phi \\ -\sinh\phi & \cosh\phi \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix}$$

Boosts are rotations in the (ct, x) plane with hyperbolic angle ϕ.

Rapidity: The Natural Parameter

$$\phi = \tanh^{-1}(v/c)$$

Velocities do **not** add linearly:

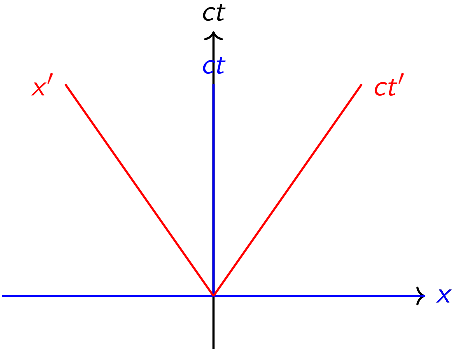
$$u' = \frac{u + v}{1 + uv/c^2}$$

But rapidities do:

$$\phi_{\text{total}} = \phi_1 + \phi_2$$

Makes group structure manifest.

Diagram: Lorentz Boost as Hyperbolic Rotation



The primed axes tilt toward the light cone as velocity increases.

Proper Time and Worldline Geometry

Spacetime arc length:

$$d\tau = \frac{1}{c}\sqrt{ds^2}$$

For a particle moving with velocity $v(t)$:

$$d\tau = dt\sqrt{1 - \frac{v^2}{c^2}}$$

Total proper time:

$$\tau = \int dt\,\sqrt{1 - \frac{v^2(t)}{c^2}}$$

The straightest worldlines (geodesics) maximize proper time.

4-Velocity and 4-Momentum

4-velocity:

$$U^\mu = \frac{dx^\mu}{d\tau} = \gamma(c,\, v_x,\, v_y,\, v_z)$$

4-momentum:

$$P^\mu = mU^\mu = \left(\frac{E}{c},\, p_x,\, p_y,\, p_z\right)$$

Invariant mass:

$$P_\mu P^\mu = m^2 c^2$$

Energy–momentum relation:

$$E^2 = p^2 c^2 + m^2 c^4$$

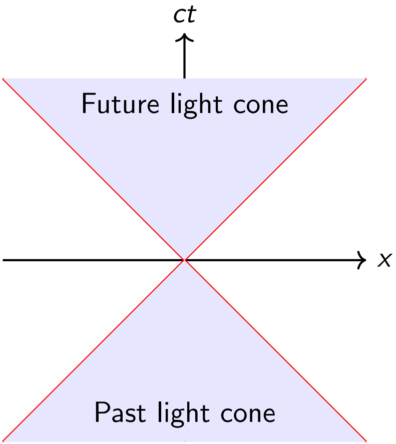
Spacetime Classification of Intervals

$$s^2 = (c\Delta t)^2 - (\Delta x)^2$$

- $s^2 > 0$: timelike (causal influence possible)
- $s^2 = 0$: lightlike
- $s^2 < 0$: spacelike (no causal connection)

Classification is Lorentz invariant.

Minkowski Diagram with Timelike / Spacelike Regions



Why Minkowski Geometry Matters

- Lorentz covariance becomes manifest: just geometry.
- Special relativity becomes a theory of rotations in spacetime.
- Proper time = arc length in Minkowski manifold.
- Foundation of:
  - General relativity
  - Quantum field theory
  - Particle accelerator physics

Conclusion

- Minkowski unified space and time into a single geometric entity.
- Lorentz transformations = hyperbolic rotations preserving the interval.
- Spacetime diagrams visualize causality and motion.
- Modern theoretical physics built directly on Minkowski's insights.

Thank you!