Lagrange and Euler and Rise of Theoretical Physics

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Introduction

- Leonhard Euler and Joseph-Louis Lagrange played a crucial role in shaping what we now call theoretical physics.
- Newton laid the groundwork for classical mechanics, but his geometric approach was difficult to apply to more complex systems.
- Euler and Lagrange addressed this challenge by introducing ideas such as variational principles and generalized coordinates.
- Their work shifted physics from geometrical approach to mathematical reasoning.
- This helped in the development of quantum mechanics, modern physics, and field theory.

Historical Background

- Newton's law was difficult to apply to more complex systems such as planetary motion, vibrating strings, or the deformation of solid materials.
- Euler introduced differential equations to describe the motion of fluids and the bending or twisting of solid bodies in a systematic way.
- Lagrange expanded these ideas to present a more general and elegant formulation of mechanics.
- Their combined efforts led to the Euler–Lagrange equation which shows that many physical processes can be understood by imagining nature selecting the path of least action.
- Euler and Lagrange helped shift physics away from purely geometric and diagram-based methods toward an energy-based framework that works in any coordinate system.
- Their contributions laid to modern theoretical physics, electromagnetism and quantum field theory.

Leonhard Euler: Mathematical Foundations of Physical Law

Euler's Analytical Approach to Physics

- Euler believed that nature can be better understood through precise and elegant mathematics.
- He helped develop differential equations, giving scientists a precise tool to describe motion and physical processes.
- He developed equations for rigid-body motion and rotation, laying foundations later used in robotics, engineering, and aerospace.
- He introduced the Euler equations for ideal fluid flow, a cornerstone of fluid dynamics that still models many low-viscosity systems.
- He also solved challenging problems—such as cases of the three-body problem and motion in complex gravitational fields—that others deemed nearly impossible.
- Through clear mathematical and practical ideas, Euler helped build the foundation of modern physics and many of the computing methods we use today.

Calculus of Variations

- In the 18th century, Johann Bernoulli developed the calculus of variations inspired by problems that involve nature to choose the most efficient path.
- His solution to the brachistochrone problem became a new way of thinking in physics and mathematics.
- Euler shaped the modern calculus of variations, introducing functional notation and creating the Euler–Lagrange equation, the variational counterpart to setting a derivative to zero.
- Euler showed that nature often follows the easiest path (principle of least action), helping develop tools for Lagrangian mechanics and modern physics.

Euler’s Equation of Motion

- Euler’s equation help us to describe how solid objects rotate, spin changes, and how the torques acting on them cause those changes.
- They also show how a body’s moment of inertia affects its motion—how hard it is to twist or rotate.
- Today, these ideas are used in controlling satellites and operating robotic arms.
- Euler’s work on rotation and fluid flow transformed physics by basing it on clear mathematical laws.
- His work influenced major breakthroughs, including Maxwell’s equations, quantum mechanics equations, and Einstein’s theory of gravity.

Joseph-Louis Lagrange: The Birth of Analytical Mechanics

Analytical Mechanics

- Lagrange replaced Newton's geometric diagrams with a fully algebraic approach, creating a major turning point in physics.
- He believed nature's laws should be expressed through math, not pictures, even declaring, "no diagrams will be found in this work".
- Using the principle of virtual work and calculus of variations, Lagrange derived motion equations by viewing physical systems as optimization problems.
- He introduced the Lagrange multiplier to handle constraints, unifying statics, dynamics, and fluid mechanics in one framework.
- His work became a foundation of modern theoretical physics and influenced nearly every major development that followed.

Generalized Coordinates and Constraints

- Lagrange's use of generalized coordinates made it much easier to describe complex mechanical systems with constraints.
- Instead of x, y, z coordinates, Lagrange used a smaller set of variables suited to a system's shape, like a bead on a wire or a pendulum's angle.
- This simplification works because holonomic constraints let us describe a system using algebraic relations between its coordinates and time.
- Lagrange used the system's constraints to choose coordinates, so he didn't need to calculate forces like tension or normal forces, since they do no work and don't appear in the Lagrangian.
- Lagrange's framework using coordinates, virtual work, and constraints simplifies even complex mechanical systems.

The Lagrangian Function and Least Action

- Lagrange introduced a new way to describe motion by focusing on a system's energy instead of its forces.
- He defined the Lagrangian, $L = T - V$, to describe motion, especially in complex or constrained systems.
- Using the principle of least action, Lagrange derived the Euler–Lagrange equation:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \left(\frac{\partial L}{\partial q_i}\right) = 0$$

This determines the system´s path by making the integral of the Lagrangian stationary.

- This became a cornerstone of modern physics, shaping classical mechanics and quantum field theory.

The Rise of Theoretical Physics

From Newtonian Forces to Lagrangian Energy Principles

- Lagrange's energy-based formulation marked a major shift, replacing Newton's force-based mechanics in theoretical physics.
- Newton's mechanics describes motion via forces ($F = ma$), but calculating constraint forces like tension or normal forces becomes tedious for complex or interconnected systems.
- Lagrange redefined motion using the Lagrangian and least action with generalized coordinates.
- This change from forces and geometry to energy and analysis made physics easier to describe universally.
- It paved the way for major breakthroughs, turning physics into a field guided by deep mathematical principles rather than case-by-case methods.

Unification of Physical Theories

- Euler and Lagrange revolutionized mechanics in the 18th century by showing that systems follow paths that make the action stationary, a concept known as the principle of least action.
- Action, defined as the integral of the Lagrangian, leads to the Euler–Lagrange equation, enabling motion analysis without computing all forces and simplifying complex systems with generalized coordinates.
- Their approach went beyond mechanics, showing that Fermat's principle in optics—light following the path of least time—is mathematically analogous to the principle of least action.
- The same variational principle applies to waves, vibrations, and even planetary motion, allowing their behavior to be described without complex force-based calculations.
- In short, Euler and Lagrange created a unified mathematical framework that forms the foundation of modern physics.

Emergence of Modern Theoretical Physics

- Joseph-Louis Lagrange's contributions have been truly foundational, leaving a profound and lasting influence on the development of modern theoretical physics.
- By formalizing the principle of action, Lagrange provided a framework, still central to physics, where modern theories start by defining an action

$$S = \int L \, dt$$

to derive a system's equations.
- Based on Lagrange's insights many major fields gradually developed: Hamiltonian mechanics, Statistical mechanics, Electromagnetic field theory, General relativity, and Quantum mechanics.

Case Studies: How Their Ideas Still Shape Physics

Example: Deriving the Pendulum via Variational Principle

- A simple pendulum shows how the Lagrangian method studies motion through energy, not forces.
- The Lagrangian of a simple pendulum can be written as:

$$L = T - V = \frac{1}{2} m l^2 \dot{\theta}^2 - mgl(1 - cos\theta)$$

- By applying the Euler–Lagrange equation, the equation of motion for the simple pendulum is:

$$\ddot{\theta} + \frac{g}{l} sin\theta = 0$$

- This shows that a single principle can describe everything from simple pendulums to quantum and relativistic systems without directly computing forces.

Example: Least Action in General Relativity

- General relativity is a clear example of how fundamental Lagrangian principles are in modern physics.
- Einstein's field equations were derived using the least-action principle applied to the Einstein–Hilbert action, rather than being guessed or based on known forces.

$$S = \int (R\sqrt{-g})d^4x$$

- These components make the action a scalar, and varying it—just as Lagrange did in classical mechanics—yields Einstein's field equations.
- This connection shows that the structure of general relativity originates from the same mathematical idea that once described simple systems like a pendulum.
- Ultimately, the least-action principle grew from a tool for mechanics into a cornerstone of modern theoretical physics.

Conclusion

- Euler and Lagrange shaped modern theoretical physics by strengthening and unifying Newton's laws with mathematical clarity.
- By introducing generalized coordinates and the principle of least action, they developed the Euler–Lagrange equations, foundational in both classical and modern physics.
- Their work enabled systematic analysis of complex systems, making physics grounded in mathematical reasoning and symmetry.
- Euler and Lagrange's ideas continue to guide physics, revealing nature's truths through the language of mathematics.

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