Nonlinear Phenomena and Chaos in Physics

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- “Chaos” - Something that is unpredictable, but not left to chance.
    - A *chaotic system* appears random, but there are deterministic rules governing its behavior.

- “Nonlinear” - Describing a function whose output is not proportional to its input.
    - Why is a parabola not chaotic?

We can all think of some examples of nonlinear functions

- $y = x^2$
- $y = e^{ax}$
- $y = Ax'' - Bx' + x$
- $y = \sin(x)$

etc . . .

Specifically, a *linear function* has a graph that is a line.
A *non*linear function is more complicated.

But what makes it *chaotic*?

Where chaos appears

- A parabola is not chaotic because it is only a nonlinear function. Chaos appears in nonlinear *systems*.
- A system, by definition, requires more than just one function.
- A linear systems can be represented by a single matrix. *Non*linear systems cannot be conveyed so easily.

Stroboscopic Maps

- Nonlinear systems are described *phenomenologically*. We look at the observed (calculated) behavior rather than the underlying rules.
- A common way of doing this is to look at the state of the system at regular intervals, usually at the same point in each period cycle.
- We also track how the same points in time differ with differing parameters of the system.
- A graph of such sampling is called a *stroboscopic map*.

Example - The Logistic Equation

This function shows a time series of a population's size over time:

$$x_{n+1} = R * x_n * (1 - x_n)$$

Where *n* is the time period, x_n is the population at that time period, and *R* is the "growth parameter".

Some Code . . . . . .

logisticEqn.py

logisticEqn.py > ...

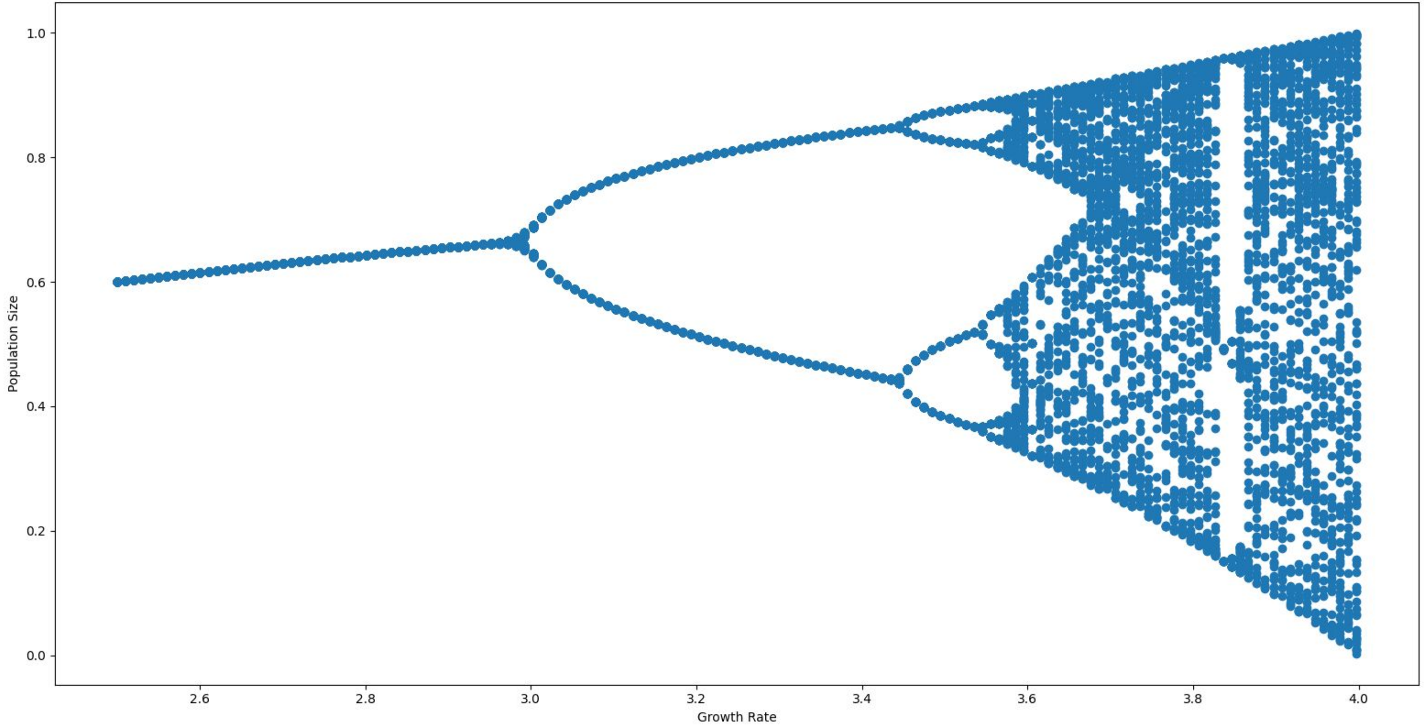
```python
import numpy as np
import matplotlib.pyplot as plt
#Includes will go here as I need them.

def logisticEqn(R, iters, popsize, maxpop):
    def newpoint(R, prevpoint):
        return R*prevpoint*(1-prevpoint)

    pop_array = np.empty(iters)
    pop_array[0] = popsize/maxpop
    for i in range(1,iters):
        pop_array[i] = newpoint(R, pop_array[i-1])

    return pop_array

```

logisticEqn.py   Logistic_script.py

Logistic_script.py > ...

```python
import numpy as np
import matplotlib.pyplot as plt
from logisticEqn import logisticEqn

Rs = np.linspace(2.5, 4.5, num=200)
start_pop = 100
maximum_pop = 10000
t_steps = 200

first_array = logisticEqn(Rs[0], t_steps, start_pop, maximum_pop)
logi_array = first_array[100::]

for i in Rs[1::]:
    new_array = logisticEqn(i, t_steps, start_pop, maximum_pop)
    logi_array = np.append(logi_array, new_array[100::], axis=0)

R_array = np.array([])
for i in Rs:
    R_array = np.append(R_array, [i]*100)

plt.scatter(R_array, logi_array)
plt.xlabel("Growth Rate")
plt.ylabel("Population Size")
plt.show()
```

The Results . . .



1.0
0.8
0.6
0.4
0.2
0.0
Population Size
2.6
2.8
3.0
3.2
3.4
3.6
3.8
4.0
Growth Rate

What was that actually a graph *of*?

- The graph is a scatterplot of multiple simulations of the logistic equation, plotted against varying growth coefficients.
- Each population level is a result of a quadratic equation, but the y is the next x, which is itself the argument to the next equation.
- We're essentially seeing cross-sections of a family of parabolas, rather than tracing a single parabola over a steady domain.
- … and we're also varying the growth coefficient.

Why was the coin-flipping game actually a bad example?

What does any of this have to do with physics?

- There are multitudes of physical systems that can’t be consistently described in closed forms.
- Many of these are nonlinear systems that exhibit chaotic behavior.
- For example . . .

The Damped, Driven Pendulum

- A pendulum subject to a damping and driving force.
- Begin with the torque:
  - $N = I\theta'' = -b\theta' - mgl*\sin(\theta) + N_d*\cos(\omega_d t)$
  - I: moment of inertia
  - b: damping coefficient
  - N_d: driving torque
  - ω_d: angular frequency of driving torque
- After some algebra, the pendulum can be described in terms of its angular position:

The Damped, Driven Pendulum

- $\theta'' = -c\theta' - \sin(\theta) + F*\cos(\omega\tau)$
  - c: new damping coefficient: $b/(ml^2\omega_0)$
  - F: magnitude of driving force: $N_d/(mgl)$
  - τ: dimensionless time: $t*(g/l)^{½}$
  - ω: driving angular frequency: ω_d/ω_0
  - ω_0: dimensionless frequency: $(g/l)^{½}$

- Even with this simplification, the system's equation is very nonlinear.

The Damped, Driven Pendulum

- This equation must be further divided into two first-order ODEs, the solutions of which must be found numerically.
- For certain values of the driving force, the graphs of the position with respect to time show no clear pattern.
- Graphs of the phase space end up looking like filled-in shapes.
- Henri Poincaré plotted a third dimension in the system's phase space ($y=\theta'$, $x=\theta$, $z=\omega\tau$), and found where the phase path intersects with regularly-spaced planes perpendicular to the z-axis, projected onto the x-y plane. This is a stroboscopic map called a *Poincaré section*.

The Damped, Driven Pendulum

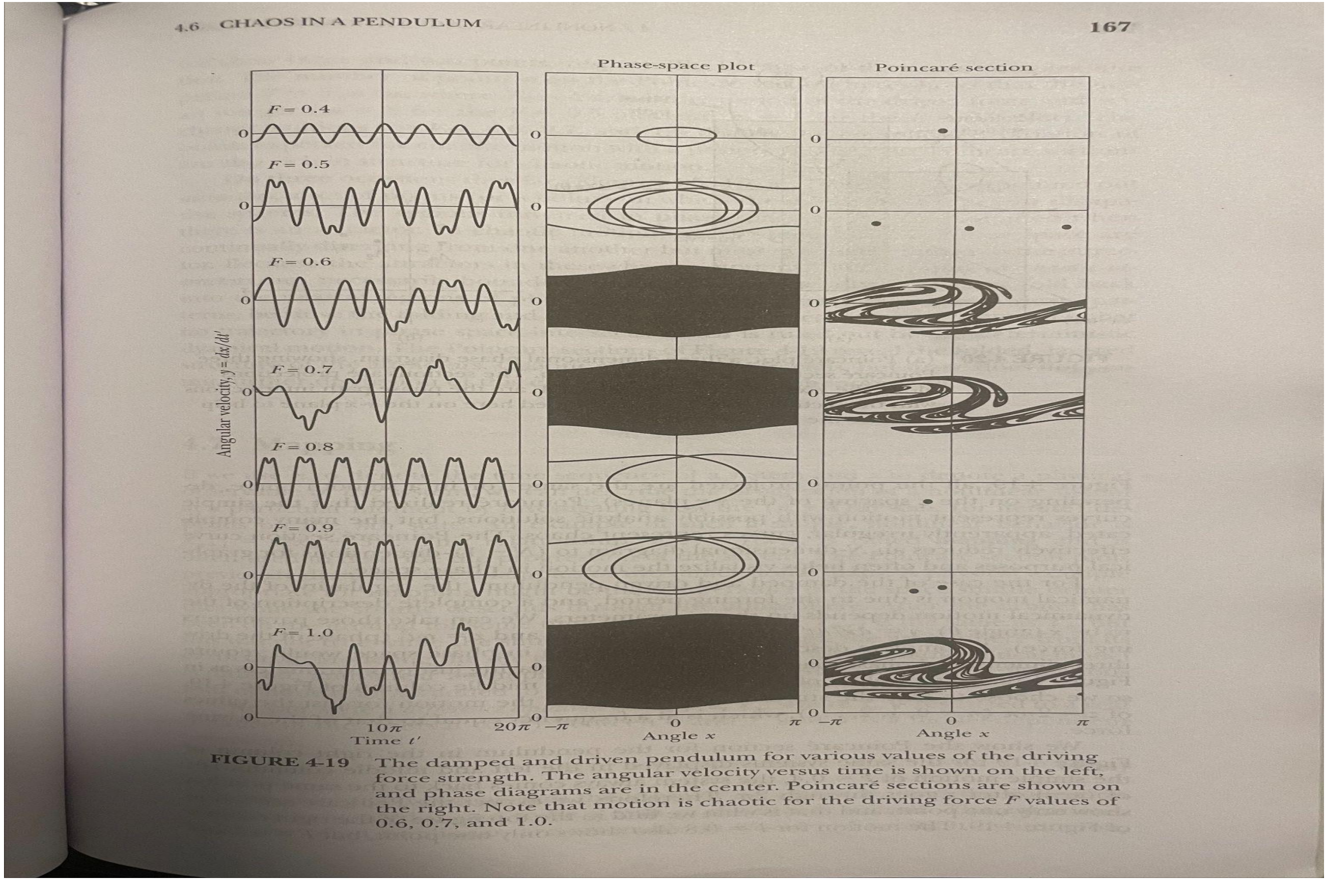


FIGURE 4-19 The damped and driven pendulum for various values of the driving force strength. The angular velocity versus time is shown on the left, and phase diagrams are in the center. Poincaré sections are shown on the right. Note that motion is chaotic for the driving force F values of 0.6, 0.7, and 1.0.

From Thornton and Marion's "Classical Dynamics of Particles and Systems" 5th ed.

Can there be chaos in quantum systems?

Quantum Chaos

- In general, quantum systems are based on combinations of the Schrodinger equation, which is linear.
  - So, no. Quantum systems don’t possess the nonlinearity required for chaotic behavior.

- However, quantum mechanics must, when expanded in scale, be able to reproduce the behavior of classical systems.
  - So how does this work with the possibility of chaos in those classical systems?

- The study of this type of correspondence is called “quantum chaos”.

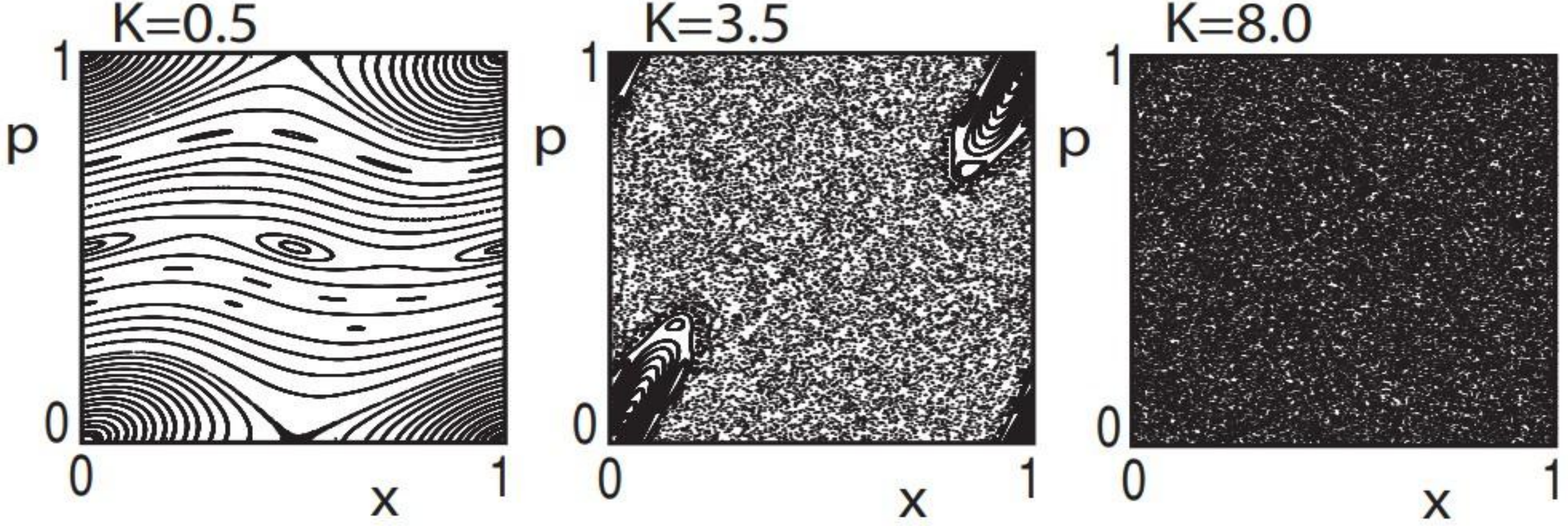
The Classical Kicked Rotor

- Imagine a particle constrained to move in a circle. At regular time intervals, it is given a "kick" in which a force is applied instantaneously. This causes it to move.
- If the circumference of the circle is 1 (radius of $1/(2\pi)$), and the period between each kick is also 1, we can write a general Hamiltonian for the system:
  - $H(x,p) = p^2/2 + V(x) * \sum_{n=-inf}^{inf} \delta(t-n)$
- If we sample the position and momentum of this system just before each kick, we can have a stroboscopic map in which the successive points are related:
  - $p_{n+1} = p - V'(x_n)$ and $x_{n+1} = x_n + p_{n+1}$
  - Note the similarity to the Logistic equation.

The Classical Kicked Rotor

- The simplest potential *V* is a multiple of a cosine:
  - $V(x) = -[K/(4\pi^2)] * \cos(2\pi x)$
- This gives our “simplest Hamiltonian”:
  - $H(x,p) = p^2/2 + -[K/(4\pi^2)] * \cos(2\pi x) * \sum_{n=-inf}^{inf} \delta(t-n)$
- Different values of *K* lead to vastly different forms of behavior, much like the growth coefficient of the Logistic Equation.

Poincaré Sections of the Classical Kicked Rotor



From “Introduction to Quantum Chaos” by Denis Ullmo and Steven Tomsovic, University of Paris-Sud, republished by Pullman of Washington State University, 2014

The Quantum Kicked Rotor

- We can represent the time evolution of a quantum-scale kicked rotor through a time-step propagator *U*:
    - $\psi(x; t=n+1) = U\psi(x; t=n)$

- Where the expression of the operator *U* is:
    - $\langle m|U|m'\rangle = (iM)^{-1/2}\ e\hat{}(i\pi(m-m')^2/M)\ *\ e\hat{}(i[KM/2\pi]*\cos(2\pi(m+1/2)/M)$

    - Where *m* indicates an allowed discrete position state, up to *M* possible states, and *K* is the same kicking parameter for the classical case.

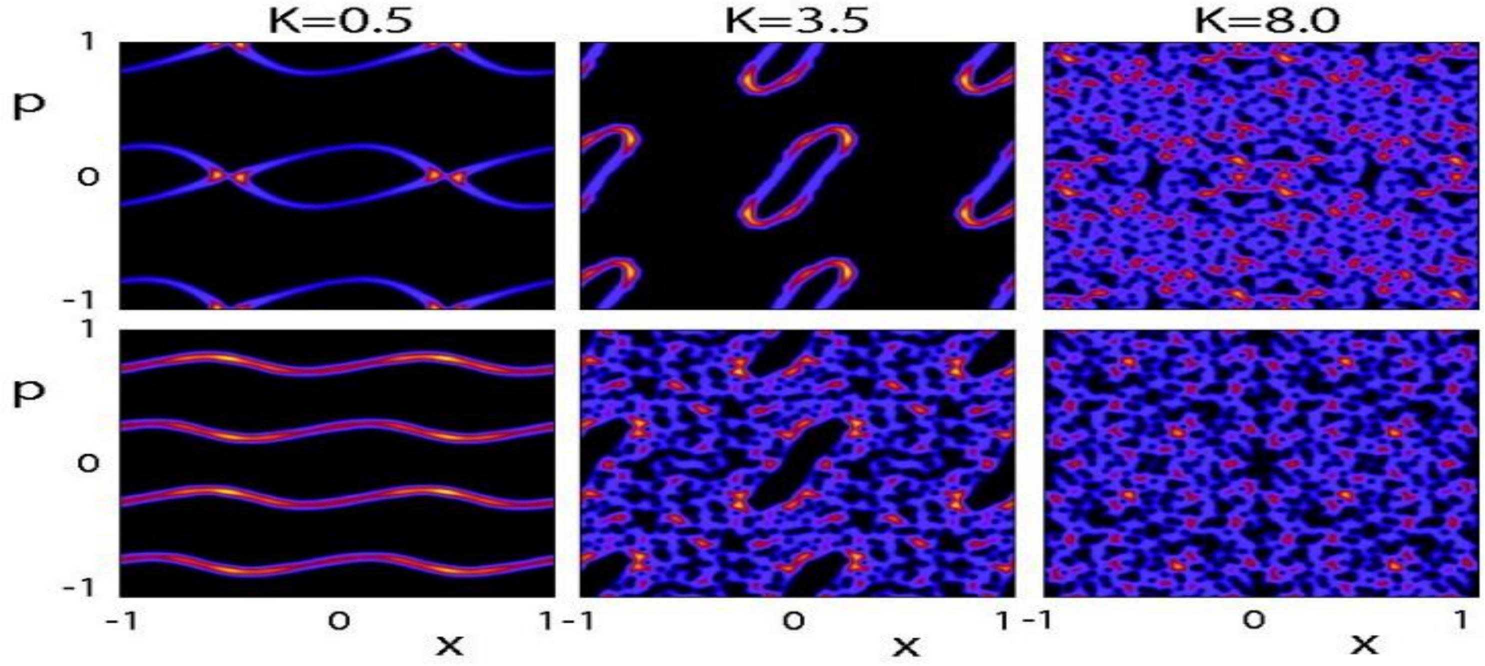
The Quantum Kicked Rotor

- The wave functions evolve in time based on the state of the previous time step.

- Each step gives us eigenvalues representing stationary states in the position basis.

- If these eigenvalues are represented using the *Husimi function*, we can see their sections in phase space:

The Quantum Kicked Rotor



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In Conclusion

- “Chaos” describes phenomena that are unpredictable, yet entirely deterministic.
- These arise from systems of multiple nonlinear functions, either seen all at once or over discrete time steps, each representing a slightly different function.
- Chaotic behavior can arise from seemingly simple physical systems if the parameters are right.
- Even quantum systems can exhibit chaotic behavior in similar regimes to their corresponding classical counterparts.

References and Sources

- Eli Tziperman, “Chaos Theory: A Brief Introduction”, ‘Chaos and Weather Prediction’, Harvard University
- Denis Ullmo and Steven Tomosovic, “Introduction to Quantum Chaos”, University of Paris-Sud, republished by Pullman at the Washington State University, 2014
- Thornton and Marion, “Classical Dynamics of Particles and Systems”, 5th ed.
- Garnett Williams, “Chaos Theory Tamed”, 1st ed. 1997
    - *I was going by what I remembered from this one.*

Questions?