Possible Exam Questions

1. Show that harmonic series diverge

2. Prove that $\zeta(2) = \sum_{n=1}^{\infty} n^{-2}$ is converging

3. Test the convergences of

a) $\sum_{n=2}^{\infty} (\ln n)^{-1}$

(b) $\sum_{n=1}^{\infty} \frac{n!}{10^n}$

(c) $\sum_{n=1}^{\infty} \frac{1}{2n(2n+1)}$

(d) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n(n+1)}}$

(e) $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

(f) $\sum_{n=1}^{\infty} \frac{1}{n2^n}$

(h) $\sum_{n=1}^{\infty} \ln\left(1 + \frac{1}{n}\right)$

4. Prove the Leibniz criterion of convergence for alternating series

5. Obtain Taylor series for $\ln(1+x)$ function and find the range of convergence

6. Prove the uniqueness theorem for power series.

7. Express sin (x) and cos (x) functions through the power series. (*(Exercise 1.2.8)*)

8. Using Levi - Civita representation of cross product prove (a) $\vec{A}\times\vec{B} = -\vec{B}\times\vec{A}$

(b) $\vec{A}\cdot\left(\vec{B}\times\vec{C}\right) = \vec{B}\cdot\left(\vec{C}\times\vec{A}\right) = \vec{C}\cdot\left(\vec{A}\times\vec{B}\right)$

(c) $\vec{A}\times\left(\vec{B}\times\vec{C}\right) = \vec{B}\left(\vec{A}\cdot\vec{C}\right) - \vec{C}\left(\vec{A}\cdot\vec{B}\right)$

9. Calculate the gradient of $f(r)\, r^n$ and consider cases of $f = 1$, $n = 1$, and $f = 1$ and $n = -1$.

10. Calculate divergence of $f(r)\, r^n\, \hat{r}$ ases of $f = 1$, $n = 1$, and $f = 1$ and $n = -2$.

11. Calculate curl of $f(r)\, r^n\, \hat{r}$.

12. Calculate curl of $-z\,\hat{e}_x + x\,\hat{e}_y$

13. Calculate $\vec{\nabla}\cdot\vec{\nabla}\,\phi$ and then consider case of $\phi = r^n$

14. Calculate $\vec{\nabla}\times\vec{\nabla}\,\phi$ and $\vec{\nabla}\cdot\left(\vec{\nabla}\times\vec{V}\right)$

15. Simplify $\vec{\nabla}\times\left(\vec{\nabla}\times\vec{V}\right)$ using Levi - Civita constants.

16. Calculate $\vec{\nabla}\cdot\left(f(r)\,\vec{V}\right)$ and $\vec{\nabla}\times\left(f(r)\,\vec{V}\right)$

17. Show that $\vec{\nabla}\left(\vec{A}\cdot\vec{B}\right) = \left(\vec{B}\cdot\vec{\nabla}\right)\vec{A} + \left(\vec{A}\cdot\vec{\nabla}\right)\vec{B} + \vec{A}\times\left(\vec{\nabla}\times\vec{B}\right) + \vec{B}\times\left(\vec{\nabla}\times\vec{A}\right)$

18. Show that

$$\int \vec{A}(r) \cdot \vec{\nabla} f(r)\, d^3 r = -\int f(r) \left(\vec{\nabla} \cdot \vec{A}(r)\right) d^3 r$$

$$\int \vec{V}(r) \cdot \left(\vec{\nabla} \times \vec{A}(r)\right) d^3 r = \int \vec{A}(r) \cdot \left(\vec{\nabla} \times \vec{V}(r)\right) d^3 r$$

19. Scetch the Gauss Theorem

20. Calculate $\vec{\nabla} \cdot \left(f(r)\, \vec{V}\right)$ and $\vec{\nabla} \times \left(f(r)\, \vec{V}\right)$

21. Show that $\vec{\nabla}\left(\vec{A} \cdot \vec{B}\right) = \left(\vec{B} \cdot \vec{\nabla}\right) \vec{A} + \left(\vec{A} \cdot \vec{\nabla}\right) \vec{B} + \vec{A} \times \left(\vec{\nabla} \times \vec{B}\right) + \vec{B} \times \left(\vec{\nabla} \times \vec{A}\right)$

22. Show that

$$\int \vec{A}(r) \cdot \vec{\nabla} f(r)\, d^3 r = -\int f(r) \left(\vec{\nabla} \cdot \vec{A}(r)\right) d^3 r$$

$$\int \vec{V}(r) \cdot \left(\vec{\nabla} \times \vec{A}(r)\right) d^3 r = \int \vec{A}(r) \cdot \left(\vec{\nabla} \times \vec{V}(r)\right) d^3 r$$

23. Scetch the Gauss Theorem

24. Show that

$$\oint_{\delta V} F(r)\, d\vec{\sigma} = \int_V \vec{\nabla} F(r)\, d^3 r$$

$$\oint_{\delta V} d\vec{\sigma} \times \vec{V}(r) = \int_V \vec{\nabla} \times \vec{V}(r)\, d^3 r$$

25. Scetch the Stokes' theorem

26. Show that

$$\int_S d\vec{\sigma} \times \vec{\nabla} F(r) = \oint_{\delta S} F(r)\, d\vec{r}$$

$$\int_S \left(d\vec{\sigma} \times \vec{\nabla}\right) \times \vec{V}(r) = \oint_{\delta S} d\vec{r} \times \vec{V}(r)$$

27. Express Maxwell equations through scalar and vector potentials using Lorentz gauge

28. Derive Gauss' Law

29. Show that $\nabla^2 \left(\frac{1}{r}\right) = -4 \pi \delta(r)$

30. Calculate

$\vec{\nabla}\phi$

$\vec{\nabla} \vec{V}$

$\nabla^2 \phi$

$\vec{\nabla} \times \vec{V}$

for general case of coordinate represntation in 3 space.

31. Calculate above expressions for Cylindrical reference frame

32. Calculate above expressions for Spherical refernce frame

33. Calculate operators of problem 1, for the case of $\phi(r)$, $\vec{V} = \vec{r} B(r)$. At the end consider the spacial cases of $\phi(r) = r^n$ and $B(r) = r^n$.

34. Calculate the following functions

$\sin^{-1}(z)$, $\sinh^{-1}(z)$

$\tan^{-1}(z)$, $\tanh^{-1}(z)$

$\cos^{-1}(z)$, $\cos^{-1}(z)$

35. Prove that

$$\sum_{n=0}^{N-1} \cos(nx) = \frac{\sin(Nx/2)}{\sin(x/2)} \cos(N-1)\frac{x}{2}$$

36. Find the analytic function

$w(z) = u(x, y) + i\,v(x, y)$

if

(a) $u(x, y) = x^3 - 3xy^2$

(b) $u(x, y) = e^{-y}\cos(x)$

37. For following $f(z)$ functions calculate $f'(z)$ and identify the maximal region within which $f(z)$ is analytic

(a) $f(z) = \frac{\sin(z)}{z}$

(b) $f(z) = \frac{1}{z(z+1)}$

(c) $f(z) = \tan(z)$

(d) $f(z) = e^{-1/z}$

38. Calculate $\oint_C z^n\,dz$ for integers $n \geq -1$

39. Prove Cauchy's Integral Theorem

40. Calculate

$\oint_C \frac{dz}{z^2 + z}$ for circle C defined by $|z| = R > 1$

41. Show that

$\oint_C z^{m-n-1}\, dz$ were m and n are integers is

Kronecker δ_{mn}

42. Evaluate

$\oint_C \frac{e^{iz}\, dz}{z^3}$ for a contour around 0.

43. Evaluate

$\oint_C \frac{\sin^2 z - z^2}{(z - a)^3}\, dz$, where the contour encircles a

44. Evaluate

$\oint_C \frac{dz}{z\,(2\,z + 1)}$ when C is a unit circle

45. Evaluate

$\oint_C \frac{dz}{z\,(2\,z + 1)^2}$ when C is a unit circle

46. Derive Taylor Expansion

47. Derive Laurent Series

48. Expand $\frac{1}{z(1-z)}$ into Laurent Series

49. Calculate Taylor expantion of $\ln(1+z)$

50. Show that $\sin(z)$ has essential singularity at infinity

51. Calculate following *integrals* :

$$I = \int_0^{2\pi} \frac{d\theta}{1 + a\cos\theta} \quad |a| < 1$$

$$I = \int_0^{2\pi} \frac{\cos 2\theta}{5 - 4\cos\theta} d\theta \quad |a| < 1$$

52. Calculate following *integrals* :

$$I = \int_0^{\infty} \frac{dx}{1 + x^2}$$

$$I = \int_0^{\infty} \frac{\cos x}{1 + x^2} dx$$

$$I = \int_0^{\infty} \frac{\sin x}{x} dx$$