Lecture6

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Lagrangian of Yang-Mills Theory

- For YM Theory one receives an invariance with respect to a local gauge SU(N) transformation

- This can be generalized for any other NonAbelian Theory.

- Back to QED
  Local gauge transformation invariance is realized by generalizing ∂_μ to the covariant derivative $D_\mu = \partial_\mu + iqA_\mu$

  and considering U(1) transformation
  $$U(1) = e^{i\omega(x)}$$

- Consider Particles (spin $=\frac{1}{2}$) that have N components $\psi = \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_N \end{pmatrix}$

  (Note that each ψ_i is a Dirac Bispinor)

- How to include interaction that can mix this components

  For this we need some (NxN) Matrix

  then $\bar{\psi}\,\hat{F}\,\psi = \bar{\psi}\psi'$

when ψ' – result of mixing the N-components of ψ

But according to the discussion of the previous lecture only $\hat{P} = \sum_{a=1}^{N^2-1} A^a_\mu(x) t^a$

Thus the interaction will have a structure $\bar{\psi} g A^a_\mu(x) t^a \psi$

where g is some universal constant for all N components

— D_μ for YM Theory will become

$$\boxed{D^{ij}_\mu = \partial_\mu \delta^{ij} + i g \sum_{a=1}^{N^2-1} A^a_\mu(x) \left(t^a\right)_{ij}}$$

— The Part of the Lagrange Density for ψ will be

$$\mathcal{L}_{\psi_A} = \sum_{i,j=1}^{N_c} \sum_{\alpha,\beta=1}^{4} \bar{\psi}_{\beta, j} \left(i \left(\gamma^\mu\right)_{\beta\alpha} D_{\mu, ij} - m \delta_{\beta\alpha} \delta_{ji} \right) \psi_{\alpha, i}$$

— If there is f- flavors of ψ then one obtains

$$\mathcal{L}_{\psi_A} = \sum_f \sum_{i,j=1}^{N_c} \sum_{\alpha,\beta=1}^{4} \bar{\psi}_{f,\beta,j} \left[i \left(\gamma^\mu\right)_{\beta\alpha} D^{(A)}_{\mu, ij} - m_f \delta_{\beta\alpha} \delta_{ji} \right] \psi_{f,\alpha,i}$$

- To obtain the Full Lagrangian one needs to add tensor Due to the interacting field

$$A^{\mu} \longrightarrow F_{\mu\nu,a}$$

(recall in QED $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$)

$$-\frac{1}{4}\sum_{\mu,\nu=0}^{3}\sum_{a=1}^{N^2-1} F_{\mu\nu,a}\,F^{\mu\nu}_{a}$$

- Need to figure out $F_{\mu\nu,a}$

- Consider the Local Gauge Invariance for $\mathcal{L}_{4A}$ with respect to $\psi' = e^{i\omega_a t_a}\psi$

$$\mathcal{L}'_{4A} = \bar{\psi}_{\alpha j}\, e^{-i\omega_b t_b}\left(i\gamma^{\mu}_{\beta\alpha}\left(\partial_\mu + i g A'^{\mu}_{a} t^{a}\right) e^{i\omega_c t_c}\psi\right)$$

$$- \bar{\psi}_{\beta j}\, m\,\delta_{\beta\alpha}\,\delta_{ji}\,\psi_{\alpha i} \to \text{ok}$$

- Consider $\boxed{e^{i\omega_c t_c}\psi = (1 + i\omega_c t_c)\psi}$

$$\mathcal{L}'_{4A} = \bar{\psi}_{\beta j}\, S^{\dagger}\left(i\gamma^{\mu}_{\beta\alpha}\left[(i\partial_\mu \omega_c t_c)\,\psi + S\,\partial_\mu \psi\right.\right.$$

$$+ig A'^{\mu}_a t^a \cdot S\psi) \quad + \text{OK}$$

– In QED we need to transform

$$A^\mu \longrightarrow A^\mu - \frac{1}{q}\partial_\mu w$$

In Analogy

$$A'^{\mu}_a t^a \longrightarrow A^\mu_a t_a - \frac{1}{g}\partial_\mu w_c t_c$$

$$\mathcal{L}'_{\psi_A} = \overline{\psi}_{\beta j} S^+ \Big(i\gamma^\mu \Big[\big(i\partial_\mu \cancel{w_c t_c} \big)\psi + \overbrace{(1+wt)}^{S}\partial_\mu \psi \Big)$$

$$+ ig A^\mu t (1+iwt)\psi - i\partial_\mu w_e t_e \cancel{(1+wt)\psi}$$

$$+ig\, i(w t_c) A^\mu t\, \psi \quad - ig (iwt) A^\mu t\, \psi$$

$$= \overline{\psi}_{\beta j} S^+ \Big(i\gamma^\mu \Big(S\,\partial_\mu \psi + \underbrace{(1+iwt)}_{S}\, ig A^\mu t$$

$$- g \big((A^\mu_a t_a)(w_b t_b) - (w_b t_b)(A^\mu_a t_a) \big)\psi \Big) =$$

$$= \overline{\psi}_{\beta i j} S^+ \Big(i\gamma^\mu \Big(S D_\mu \psi - g A^\mu_a w_b [t_a t_b - t_b t_c]\psi \Big) \Big)$$

$$= \overline{\psi}_{\beta i} S^+ \Big(i\gamma^\mu \big(S D_\mu \psi - i g A^\mu_a w_b f^{abc} t_c \psi \Big)$$

this

Not invariant, $\boxed{\text{this term}}$ is a problem

– to cancell this term one needs different Gauge transformations for A_μ

$$A'^{\mu} t \to A^{\mu} t - \frac{1}{g}\partial_\mu \omega t - f_{abc}\, \omega_a A_b^{\mu} t_c$$

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– the 3 term will result in

$+ i\, \omega_a A_b^{\mu} f^{abc} t_c \psi$

$$= \bar{\psi}_{\beta j} S^{+}\Big(i\gamma^{\mu}\big(S D_\mu \psi - \cancel{i g A_a^{\mu} \omega_b f^{abc} t_c \psi}$$

$$\cancel{- i g\, \omega_a A_b^{\mu} f^{abc} t_c \psi}\Big)$$

$$= \bar{\psi}_{\beta j}\left(i\gamma^{\mu} D_\mu - m\delta_{ji}\right)\psi_i$$

$\Rightarrow$ Thus To have $\mathcal{L}_{\psi A}$ invariant One need a transformation on

$$\boxed{\psi' \Rightarrow S\psi = e^{i\omega_a(x) t_a}\psi}$$

for a

$$A'^{\mu} \Rightarrow A^{\mu} - \frac{1}{g}\partial_\mu \omega_a - f^{abc}\, \omega_a A^{\nu} \ldots$$

$$\boxed{A'^{\mu}_{a} = A^{\mu}_{a} - \frac{1}{g}\partial^{\mu}\omega_a - f^{abc}\,\omega_b A^{\mu}_{c}}$$

$\Rightarrow$ Now One needs to define $F^{\mu\nu}_{a}$ such that it is invariant with respect to the above transformation

- In QED $F^{\mu\nu} = \partial_\mu A^\nu - \partial_\nu A^\mu$

- Assuming same for QCD

$$F^{\mu\nu}_a = \partial_\mu A^\nu_a - \partial_\nu A^\mu_a$$

$$F'^{\mu\nu}_a = \partial_\mu A^\nu_a - \frac{1}{g}\partial_\mu \partial_\nu \omega_a - f^{abc}\,\partial_\mu(\omega_b A^\nu_c)$$

$$- \partial_\nu A^\mu_a + \frac{1}{g}\partial_\nu \partial_\mu \omega_a + f^{abc}\,\partial_\nu(\omega_b A^\mu_c)$$

$$= \partial_\mu A^\nu_a - \partial_\nu A^\mu_a - f^{abc}\,\partial_\mu(\omega_b A^\nu_c)$$

$$+ f^{abc}\,\partial_\nu(\omega_b A^\mu_c)$$

- Not invariant

— Gauge Invariance of Gluonic Part of the Lagrangian

Need to prove that

$\boxed{-\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a}$ — is invariant with respect to the local Gauge transformation which is a unitary transformation

— We can write Eq (*) in the form $-\frac{1}{2} Tr\, F^a_{\mu\nu} t^a \cdot F^b_{\mu\nu} t^b = -\frac{1}{2} Tr \hat{F}_{\mu\nu} \hat{F}_{\mu\nu}$

using the relation $Tr\, t^a t^b = \frac{1}{2} \delta^{ab}$

— as it follows from eq(** a) transformation of the quark field is unitary transformation by $S\psi = e^{i\omega^a t^a} \psi$

— Eq (** b) also can be expressed through S in the form

$\hat{A}_\mu \rightarrow \ldots S(x) \ldots \frac{i}{g} \ldots S(x) \ldots S^{-1}(x)$

$$A_\mu'(x) = S(x) A_\mu S(x) + \frac{1}{g}[\partial_\mu S(x)] \ldots$$

- After this the Gauge invariance of the pure gluonic field mean

$$F'_{\mu\nu} = S F_{\mu\nu} S^{-1}$$

- In this case The kinetic energy part will be invariant

$$-\frac{1}{2} Tr\, \hat{F}'_{\mu\nu} \hat{F}'^{\mu\nu} = -\frac{1}{2} Tr\, S \hat{F}_{\mu\nu} S^{-1} S \hat{F}^{\mu\nu} S^{-1} =$$

$$= Tr\, \hat{F}_{\mu\nu} F_{\mu\nu}$$