10-Sep-14
11-Oct-17

QCD Lagrangian

Remember: Field Equation of Motion

$$\frac{d}{dx_\mu}\frac{\partial \mathcal{L}}{\partial \varphi_{,\mu}} - \frac{\partial \mathcal{L}}{\partial \varphi} = 0$$

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial \varphi^{\tau}_{,\mu}} - \frac{\partial \mathcal{L}}{\partial \varphi^{\tau}} = 0$$

$$\varphi^{\tau}_{,\mu} = \frac{\partial}{\partial_\mu}\varphi^{\tau}$$

Remember $\mathcal{L}_{QED} = \bar{\psi}(x)[\gamma_\mu(i\partial_\mu - qA_\mu) - m]\psi$

$$-\frac{1}{4}F_{\mu\nu}^2(x)$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

Applying Equation of motion

$$-\frac{1}{4}\partial_\sigma\left(\frac{\partial(F_{\mu\nu}\cdot F_{\mu\nu})}{\partial_\sigma \partial_\tau A_\tau}\right) - \frac{\partial \bar{\psi}\gamma_\mu(-qA_\mu)\psi}{\partial A_\tau} = 0$$

1

a+

$$\frac{\partial(F_{\mu\nu}F_{\mu\nu})}{\partial\,\partial_\sigma A_\tau} = \frac{\partial F_{\mu\nu}}{\partial\,\partial_\sigma A_\tau}\cdot F_{\mu\nu} + F_{\mu\nu}\frac{\partial F_{\mu\nu}}{\partial\,\partial_\sigma A_\tau}$$

$$= 2\frac{\partial F_{\mu\nu}}{\partial\,\partial_\sigma A_\tau}F_{\mu\nu} \quad + \qquad \partial_\nu A_\mu F_{\mu\nu} = -\partial_\mu A_\nu F_{\mu\nu}$$

$$\frac{\partial F_{\mu\nu}}{\partial\,\partial_\sigma A_\tau}F_{\mu\nu} = \frac{\partial(\partial_\mu A_\nu - \partial_\nu A_\mu)}{\partial_\sigma A_\tau}F_{\mu\nu}$$

$$= 2\frac{\partial\,\partial_\mu A_\nu}{\partial_\sigma A_\tau}F_{\mu\nu} = 2F_{\sigma\tau}$$

$$+ = 4F_{\mu\nu} \quad \text{into} \quad \partial +$$

$$-\partial_\sigma F_{\sigma\tau} + q\bar{\psi}\gamma_\tau\psi = 0$$

$$\partial_\sigma F_{\sigma\tau} = q\bar{\psi}\gamma_\tau\psi$$

$$\boxed{\partial_\mu F_{\mu\nu} = j_\nu}$$

Yang - Mills Theory

$\Psi(x) \rightarrow \psi_a(x)$ - triplet

$U(1)$ $SU(3)$

$\psi' = e^{-i\alpha(x)} \psi(x)$ $\psi_a' \rightarrow S(x) \psi_a$

$e^{-i\alpha_a T_a} \psi_a$

Cov. Derivative $D_\mu = \partial_\mu + iqA_\mu$ $D_\mu = \partial_\mu + igA_\mu^a T^a$

$\Rightarrow$ Idea of Covariant Derivatives

that if $\psi_a' \rightarrow S(x) \psi(a)$

Eq B $D_\mu \psi' \rightarrow S(x) D_\mu \psi$

Remember $S^\dagger S = 1$

Then $\bar{\psi} D_\mu \gamma^\mu \psi$ is L.G Invariant

This puts constrain on how A_μ^a should be changed that Eq B is satisfied

$$D_\nu \psi' = (\partial_\nu + igA_\nu^a T^a) S(y) \psi$$

$$= S (\partial_\mu + ig A_\nu^a T^a) \psi$$

$$S^{-1} (\partial_\mu + ig A_\nu^a T^a) S \cdot \psi = (\partial_\mu + ig A'^{\mu}_a T^a) \psi$$

$$S^{-1} \partial_\mu S \cdot \psi + \partial_\mu \psi$$

$$S^{-1} (\partial_\mu S \cdot \psi) + S^{-1} ig A_\nu^a T^a S \psi =$$

$$= \partial_\mu \psi + i g A'^{\mu}_a T^a \psi \implies$$

$$S^{-1} \partial_\nu S + ig S^{-1} A_\nu^a T^a S = ig A'^a_\nu T^a$$

$$\frac{-i}{g} S^{-1} \partial_\nu S + S^{-1} A_\nu^a T^a S = A'^a_\nu T^a$$

$$\boxed{A'^a_\nu T^a = \frac{-i}{g} S^{-1} \partial_\nu S + S^{-1} A_\nu T^a S}$$

Not that L.G inv of $D_\nu \psi$ can be formulated as

$$\boxed{D_\mu \to S D_\nu S^{+}}$$

From (+) for Global Transformation

$\partial_\mu S = 0$

$\boxed{A_\nu^a T^a = S^{-1} A_\nu T^a S}$ This is transformation for the adjoint representation

$D_\mu \cdot O^a =$

$= (\partial_\mu \delta^{ac} \pm i g f^{abc} A^b) O^c$

$\mathcal{L}_{spinor} = \bar{\Psi}_a (i\gamma_\mu D_\mu^{ab} - m)\Psi_b =$

$= \bar{\Psi}_i (i\gamma_\mu \partial_\mu \delta^{ij} - g\gamma_\mu A_\mu^a T^a_{ij} - m\delta_{ij})\Psi_j$

How to build the Kinetic Energy of gauge Field

QED

$[D_\mu D_\nu] = iq F_{\mu\nu}$

$[\partial_\mu + iqA_\mu, \partial_\nu + iqA_\nu] = \cancel{\partial_\mu A_\nu} -$

$(\partial_\mu + iqA_\mu)(\partial_\nu + iqA_\nu)\psi - (\partial_\nu + iqA_\nu)(\partial_\mu + iqA_\mu)\psi$

$$\partial_\mu \partial_\nu (\psi) + \partial_\mu (iqA_\nu \psi) + iqA_\mu \partial_\nu \psi$$

$$- q^2 A_\mu A_\nu \psi - \partial_\nu \partial_\mu \psi - \partial_\nu (iqA_\mu \psi)$$

$$- iqA_\nu \partial_\mu \psi + q^2 A_\nu A_\mu \psi$$

$$= iq\,\partial_\mu A_\nu \psi + iq\,\cancel{A_\nu \partial_\mu \psi} + \cancel{iqA_\mu \partial_\nu \psi}$$

$$- iq\,\partial_\nu A_\mu \psi - iq\,\cancel{A_\mu \partial_\nu \psi} - \cancel{iqA_\nu \partial_\mu \psi}$$

$$= iq\,F_{\mu\nu}$$

$$F_{\mu\nu} = -\frac{i}{q}[D_\mu D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu$$

For ~~QCD~~

$$F^a_{\mu\nu} T^a = -\frac{i}{g}[D_\mu D_\nu] =$$

$$= [\partial_\mu A_\nu^a - \partial_\nu A_\mu^a - g f^{abc} A_\mu^b A_\nu^c] T^a$$

$$\mathcal{L}_{YM} = -\frac{1}{4} \overset{Tr}{F^a_{\mu\nu} T^a_{ij} T^b_{ij} F^c_{\mu\nu}} =$$

$$-\frac{1}{4} F^a_{\mu\nu} F^c_{\mu\nu}$$

$$\frac{-i}{g}\left[\left(\partial_\mu + igA_\mu^a T^a\right)\left(\partial_\nu + igA_\nu^a T^a\right)\right]$$

$$= \frac{-i}{g}\Big[\partial_\mu\partial_\nu + ig\,\partial_\mu\left(A_\nu^a T^a \psi\right) + igA_\mu^a T^a \partial_\nu \psi - g^2 A_\mu^a T^a A_\nu^b T^b$$

$$-\partial_\nu\partial_\mu - ig\,\partial_\nu\left(A_\mu^a T^a \psi\right) - igA_\nu^a T^a \partial_\mu \psi + g^2 A_\nu T^b A_\mu T^a\Big]$$

$$= \frac{-i}{g}\left[ig\left[\partial_\mu A_\nu^a T^a - \partial_\nu A_\mu^a T^a\right] - g^2\left[A_\mu^a T^a A_\nu^b T^b - A_\nu^b T^b A_\mu^a T^a\right]\right]$$

$$= \frac{-i}{g}\left[ig\left[\partial_\mu A_\nu^a - \partial_\nu A_\mu^a\right]T^a - g^2 A_\mu^a A_\nu^b\left[T^a T^b - T^b T^a\right]\right]$$

$$[T_a\ T_b] = i f_{abc} T_c$$

$$= \frac{-i}{g}\left[ig\left[\partial_\mu A_\nu^a - \partial_\nu A_\mu^a\right]T^a - g^2 A_\mu^a A_\nu^b\, i f^{abc}\, T^c\right]$$

$$A_\mu^b A_\nu^c f^{abca}\, T^a$$

$$f^{abc}$$

$$= \left[\partial_\mu A_\nu^a - \partial_\nu A_\mu^a - g A_\mu^b A_\nu^c f^{abc}\right] T^a$$

DZ

7

$$F_{\mu\nu}^a T^a = [\partial_\mu A_\nu^a - \partial_\nu A_\mu^a - g^{abc} A_\mu^b A_\nu^c] T^a$$

$$\mathcal{L}_{YM} = -\frac{1}{2} \mathrm{Tr}\, F_{\mu\nu}^a T_{ij}^a F_{\mu\nu}^c T_{ji}^c = -\frac{1}{4} F_{\mu\nu}^a F_{\mu\nu}^a$$

$$\mathrm{Tr}\, T^a T^b = \frac{\delta^{ab}}{2}$$

Let us consider only ~~[illegible]~~

Gluonic Field

and discuss equation of Motion

$$\partial_\sigma \frac{\partial \mathcal{L}}{\partial \partial_\sigma A^\tau} - \frac{\partial \mathcal{L}}{\partial A^\tau} = 0$$

Considering $\mathcal{L}_{YM} = -\frac{1}{4} F_{\mu\nu}^a F_{\mu\nu}^a$.

$\cancel{\left[-\frac{1}{4}\left(\partial_\mu A_\nu^a - \partial_\nu A_\mu - g\right)\right.}$

$$\frac{\partial \mathcal{L}(F_{\mu\nu} F_{\mu\nu})}{\partial \partial_\sigma A_\tau} = 2 \frac{\partial F_{\mu\nu}}{\partial \partial_\sigma A^\tau} \cdot F_{\mu\nu}$$

$$\frac{\partial \mathcal{L}(F_{\mu\nu} F_{\mu\nu})}{\partial A_\tau} = 2 \frac{\partial F_{\mu\nu}}{\partial A_\tau} F_{\mu\nu}$$

8

2670777850

403263

Consider

$$\partial_\sigma \frac{\partial F_{\mu\nu}}{\partial \partial_\tau A^\tau} F_{\mu\nu} = \frac{\partial\left(\partial_\mu A_\nu - \partial_\nu A_\mu - g f^{abc} A^b_\mu A^c_\nu\right)}{\partial\, \partial_\sigma A^\tau} F_{\mu\nu}$$

$$= \left(\delta^{\mu\sigma}\delta^{\nu\tau} - \delta^{\nu\sigma}\delta^{\mu\tau}\right) F_{\mu\nu}^{-abc} =$$

$$\left(\partial_\sigma A^\tau - \partial_\tau A^\sigma - g f^{abc} A^b_\sigma A^c_\tau - \right.$$

$$\left. - \partial_\tau A^\sigma - \partial_\sigma A^\tau - g^{abc} A^b_\tau A^c_\sigma \right) =$$

$$+$$

$$= 2 F_{\sigma\tau}$$

First $\frac{1}{4} \partial_\sigma \frac{\partial F_{\mu\nu} F_{\mu\nu}}{\partial \partial_\sigma A^\tau} = \partial_\sigma F_{\sigma\tau}$

$\Rightarrow$ Second Part

$$\frac{1}{4} \frac{\partial F_{\mu\nu} F_{\mu\nu}}{\partial A^\tau} = \frac{1}{2}\left(\frac{\partial F_{\mu\nu}}{\partial A^\tau}\right) F_{\mu\nu}$$

$$= \frac{1}{2}\left(\frac{\partial\left(\partial_\mu A^\nu - \partial_\nu A^\mu - g f^{abc} A^b_\mu A^c_\nu\right)}{\partial A^\tau}\right) F^a_{\mu\nu}$$

$$= -g f^{abe}\, \delta^{bd}\, \delta^{\mu\sigma}\, A_\nu^c F_{\mu\nu}^a =$$

$$= -g f^{adc} A_\nu^c F_{\sigma\nu}^a = -g f^{dca} A_\nu^c F_{\sigma\nu}^a$$

$$= g f^{dca} A_\nu^c F_{\nu\sigma}^a$$

Eq. Mo. becomes

$$\partial_\sigma F_{\sigma\tau}^d - g f^{dca} A_\nu^c F_{\nu\tau}^a = 0$$

$$\text{(x)}\quad \boxed{\partial_\mu F_{\mu\nu}^a - g f^{abc} A_\mu^b F_{\mu\nu}^c = 0}$$

What is this?

$$\text{(x)}\quad D_\mu F_{\mu\nu}{}^a = \cancel{(\partial_\mu + i g A_\mu^d t^d)}\ (\partial_\mu \delta^{ac} - g f^{abc} A^b) F_{\mu\nu}^c = 0$$

$$D_\mu^{AR} = (\partial_\mu \delta^{ac} - g f^{abc} A_\mu^b) F_{\mu\nu}^c$$

$$\boxed{{}^{AR}D_\mu F_{\mu\nu}^c = 0}$$

Remember For QED

$$\boxed{\partial_\mu F_{\mu\nu} = 0}$$

D_μ^{AR}

Adjoint Representation

Norits - $D_\mu^{AR\,ac} = \partial_\mu \delta^{ca} - g f^{abc} A_\mu^b$

$$\boxed{D_\mu^{AR} F_{\mu\nu}^a = \partial_\mu F_{\mu\nu}^a - g f^{abc} A_\mu^b F_{\mu\nu}^c}$$

With the Matter Field

$D_\mu^{AR} F_{\mu\nu}^a = g \bar{\psi} \gamma_\nu T^a \psi = J_\nu^a$

QED

$$\boxed{\partial_\mu F_{\mu\nu} = J_\nu}$$

=> ~~Effective~~ conservation of Color charge

$$\boxed{D_\nu D_\mu F_{\mu\nu}^a = D_\nu J_\nu^a = 0}$$

$$D_\nu J_\nu^a = g \underbrace{\partial_\nu \bar{\psi} \gamma^\nu t^a \psi}_{= 0 \;\; D.Eq} - \underbrace{g^2 f^{abc} A_\nu^b \bar{\psi} \gamma^\nu \psi t^c}_{f^{abc} \text{ antisymmetric}}$$

In QED

$\partial_\nu J_\nu = 0$ current conservation

$$\oint_{\Theta\,\text{Infinite}} \vec{J}\cdot d\vec{S} = \int \nabla \vec{J}\, dV = \frac{d}{dt}\int J_0\, dV$$

$$0 = \frac{d}{dt} Q$$

For QCD

This does not mean current conservation

$${}_{AR}D_\mu J_\mu^a = 0 = \partial_\mu J_\mu$$

For QCD

$$\partial_\mu F_{\mu\nu}^a - g f^{abc} A_\mu^b F_{\mu\nu}^c = J_\nu^a$$

$$\partial_\mu F_{\mu\nu}^a = \underbrace{J_\nu^a + g f^{abc} A_\mu^b F_{\mu\nu}^c}$$

$$j_\nu^a = J_\nu^a + g f^{abc} A_\mu^b F_{\mu\nu}^c$$

Conserved color current

but this is not unique

12 one can add $+\, g\, \partial_\nu f^{abc} A_\nu^b A_\mu^c$

In QED, in addition to

$$\partial_\mu F_{\mu\nu} = j_\nu$$

one has

$$\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0$$

$$\partial_\mu\left(\left(\tfrac{1}{2}\right) \varepsilon_{\mu\nu\lambda\sigma} F_{\lambda\sigma}\right) = 0$$

Bianchi Identities

~~$D_\lambda F_{\mu\nu}^a + D_{\mu} F_{\nu}^a +$~~

$$D_\lambda F_{\mu\nu}^a + D_\mu F_{\nu\lambda}^a + D_\nu F_{\lambda\mu}^a = 0$$

$\boxed{\text{Ghost Fields}}$

Let us add $\Delta \mathcal{L}_{\text{constr}} = -\frac{1}{2\xi}\left(\partial_\mu A_\mu^a\right)^2$

$\boxed{\text{in QED}}$

$\partial_\mu F_{\mu\nu} = J_{em,\nu} \Rightarrow$ becomes

$$\boxed{\partial_\mu F_{\mu\nu} + \frac{1}{\xi}\partial_\nu(\partial_\alpha A_\alpha) = J_{\nu,em}}$$

$\boxed{\text{in QCD}}$

$$\partial_\mu F_{\mu\nu}^a - g f^{abc} A_\mu^b F_{\mu\nu}^c + \frac{1}{\xi}\partial_\nu(\partial_\alpha A_\alpha^a) = J_\nu$$

or

$$D_\mu^{ca} F_{\mu\nu}^c + \frac{1}{\xi}\partial_\nu \eta^a = J_\nu$$

13

$$D_\nu^{ac} D_\mu F_{\mu\nu} + \frac{1}{\xi} D_\nu^{ac} \partial_\nu \eta^c = D_\nu^{ac} J_\nu$$

$$D_\nu^{ac} \partial_\nu \eta^c = 0$$

$$\boxed{\partial^2 \eta^a - g f^{abc} A_\nu^b \partial_\nu \eta^c = 0}$$

η field interacting ...

How to solve this issue

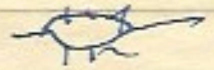
Faddeev, Popov, de Witt

$\Rightarrow$ Introduce multiplet $(\phi^+)^a$ — Faddeev Ghost

which satisfies same equation like η

$$(D_\mu \partial_\mu \phi^+)^a = \partial^2 (\phi^+)^a - g f^{abc} A_\mu^b (\phi^+)^c = 0$$

quantization like Fermions

for  (-1)

$$\mathcal{L}_{ghost} = (\partial_\mu \phi^+)^a (D_\mu \phi)^a = (\partial_\mu \phi^+)^a \partial_\mu \phi^a - g f^{abc} (\partial_\mu \phi^+)^a A_\mu^b \phi^c$$

Axial Gauge

$$\Delta \mathcal{L} = \pm \frac{1}{2\xi b^2} (b_\mu A_\mu) \partial_\sigma^2 (b_\nu A_\nu)$$

$$= -\frac{1}{2\xi b^2} \left(\partial_\sigma (b \cdot A)\right)^2$$

$$+ \partial_\sigma \delta_\xi (bA)$$

in QED

$$\partial_\mu F_{\mu\nu} - \frac{1}{\xi_1 b^2} b_\nu \partial_\sigma^2 (bA)$$

in QCD

$$\partial_\mu F_{\mu\nu}^a - g f^{abc} A_\mu^b F_{\mu\nu}^c + \frac{1}{\xi b^2} b_\nu \partial_\sigma^2 (bA^a) = J_\nu$$

$$D_\nu \quad D_\mu F_{\mu\nu}^a + \frac{1}{\xi b^2} b_\nu \partial_\sigma^2 (\eta^a) = J_\nu$$

$$D_\nu b_\nu \partial_\sigma^2 (\eta^a) = 0$$

$$\partial_\nu b_\nu \partial_\sigma^2 (\eta^a) - g f^{abc} A_\nu b_\nu \partial_\sigma^2 (\eta^a) = 0$$

$\partial_\sigma^2 (\eta a) = 0$ Solution of the above equation: