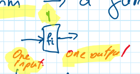
L26

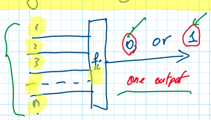
Monday, April 12, 2021 4:11 PM

The Deutsch-Jozsa Algorithm

- Deutches algorithm → a function of one variable



- Consider a function of n-variables



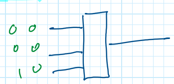
Constant: output always 1; always 0

Balanced: output half 1, half 0

- if these functions given at random
how many evaluations of these functions we need to call to define — constant / Balanced

- Example $n=3$ 2^3 inputs

$(0,0,0)$
$(0,0,1)$
$(0,1,0)$
$(0,1,1)$
$(1,0,0)$
$(1,0,1)$
$(1,1,0)$
$(1,1,1)$



- Classically - suppose we evaluate

1st { $f(0,0,0)=1$ — can not say anything

2nd { next $f(0,0,1)=0$ — Balanced!
6th if $f(0,0,1)=1$ — can not say anything

3rd { $f(0,1,0)=1$ — no

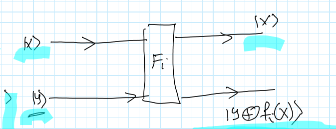
4th { $f(0,1,1)=1$ — no

5th { $f(1,1,0)=1$ — Constant!

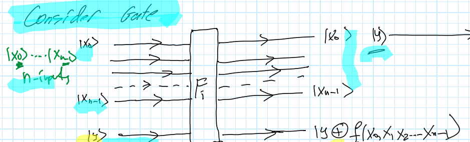
min 2 measurements, max $5 = 2^{n-1}+1$

- for $n \gg 1$ $2^{n-1}+1 \sim e^{n}$ — impossible number of classical measurements

- Deutsch-Jozsa Scheme



Consider Gate

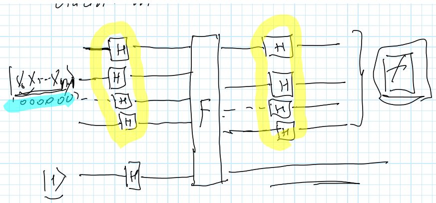


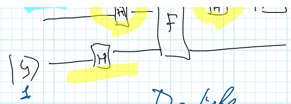
$i = n.$

- check
$|x_0, x_1 \cdots x_{n-1}\rangle \otimes |0\rangle \longrightarrow |x_0, x_1 \cdots x_{n-1}\rangle \otimes |0 \oplus f(x_0 \cdots x_{n-1})\rangle = |x_0, x_1 \cdots x_{n-1}\rangle \otimes |f_i(x_0 \cdots x_{n-1})\rangle$
$|x_0, x_1 \cdots x_{n-1}\rangle \times |1\rangle \longrightarrow |x_0, x_1 \cdots x_{n-1}\rangle \otimes |1 \oplus f(x_0 \cdots x_{n-1})\rangle = |x_0, x_1 \cdots x_{n-1}\rangle \otimes |f(x_0 \cdots x_{n-1}) \oplus 1\rangle$

!

- Quantum Circuit





Duetche

The Kronecker Product of Hadamard Matrices

⇒ Matrix for Hadamard gate is given

$|0\rangle = \binom{1}{0}$ $|1\rangle = \binom{0}{1}$

This follows from

$H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

$H|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$

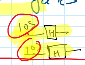
$\hat{H} = ?$

$|0\rangle = \binom{1}{0}$ $|1\rangle = \binom{0}{1}$

$\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \frac{1}{\sqrt{2}}\binom{1}{1} = \frac{1}{\sqrt{2}}\binom{1}{0} + \frac{1}{\sqrt{2}}\binom{0}{1}$

$\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$

⇒ Suppose we input two qubits and send through H gates

$H(|0\rangle|0\rangle) =$ 

H( )

H( )

H( )

⇒ We can rewrite this in terms of 4-d kets

Example $|0\rangle \otimes |0\rangle = \binom{1}{0} \otimes \binom{1}{0} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$

$|0\rangle \otimes |1\rangle = \binom{1}{0} \otimes \binom{0}{1} = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$

$|1\rangle \otimes |0\rangle = \binom{0}{1} \otimes \binom{1}{0} = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}$

$|1\rangle \otimes |1\rangle = \binom{0}{1} \otimes \binom{0}{1} = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$

4d

$H|0\rangle \otimes |0\rangle = H_1^{\otimes 2}\begin{pmatrix}1\\0\\0\\0\end{pmatrix} = \frac{1}{\sqrt{2}}(|00\rangle + |01\rangle + |10\rangle + |11\rangle) = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$

$H|0\rangle \otimes |1\rangle = H_4 \begin{pmatrix} 1 \\ [illegible] \\ 0 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\left(|00\rangle - |01\rangle + |10\rangle - |11\rangle\right) = \frac{1}{2}\begin{bmatrix} [illegible] \\ [illegible] \\ 1 \\ -1 \end{bmatrix}$

$H|1\rangle \otimes |0\rangle = H_4^{\otimes 2} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \cdots \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$

$H|1\rangle \otimes |1\rangle = H_4^{\otimes 2} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \cdots \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ -1 \\ 0 \end{bmatrix}$

— Solve for H in 4d.

⇒ For 3-qubit system

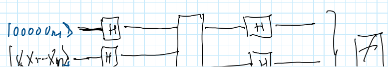
⇒ For n-qubit system

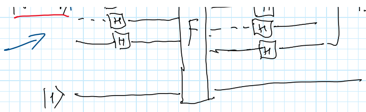
Recursive Formula
Kronecker Products

⇒ Each row is proportional to $\left(\frac{1}{\sqrt{2}}\right)^n$

Same for the state

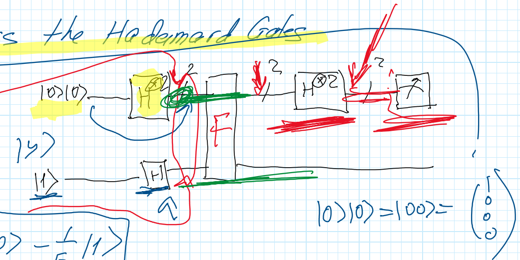
⇒ Return to Deutsch-Jozsa Algorithm





$\Rightarrow$ Consider case of $n=2$

Step. 1 The Qubits Pass the Hadamard Gates



$-|y\rangle$ entyr $|1\rangle$ $\quad H|1\rangle = \boxed{\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle}$

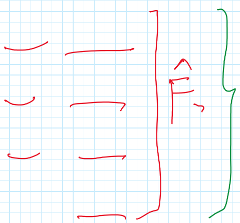
(The above calculation works for every n
After n-qubits have passed $H^{\otimes n}$ they are
in superposition of all possible states
each having same probability amplitude $\left(\frac{1}{\sqrt{2}}\right)^n$

Step 1

- So for $n=2$ we have after $H^{\otimes 2}$ and H (for y)

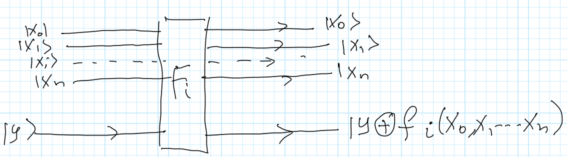
Input after Hadamards

$$\frac{1}{2}\left(|00\rangle + |01\rangle + |10\rangle + |11\rangle\right) \otimes \left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right) =$$



Entering To F_i Gate

Step 2: Remember the Gate

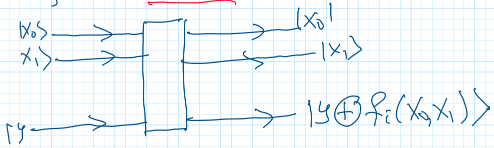


Input $\qquad$ Output

$|x_0\rangle|x_1\rangle \dots |x_n\rangle \otimes |0\rangle \longrightarrow |x_0\rangle|x_1\rangle \dots |x_n\rangle \otimes |f_i(x_1 \dots x_n)\rangle$

$|x_0\rangle|x_1\rangle\cdots|x_n\rangle \otimes |1\rangle \longrightarrow |x_0\rangle|x_1\rangle\cdots|x_n\rangle\otimes|f_i(x_1\cdots x_n)\oplus 1\rangle$

for $n=2$



input

$|x_1\rangle|x_2\rangle\otimes|0\rangle \to |x_0\rangle|x_1\rangle\otimes|f_i(x_1 x_2)\rangle$

$|x_1\rangle|x_2\rangle\otimes|1\rangle \to |x_0\rangle|x_1\rangle\otimes|f_i(x_1 x_2)\oplus 1\rangle$

!!

Using this

Using the fact that if a is either 0 or 1

$$|a\rangle - |a\oplus 1\rangle = (-1)^a(|0\rangle - |1\rangle)$$

$a \to f_i$

$\Rightarrow$

One obtains

(a) –
(b) –
(c) –
(d) –

– Bottom qubit is not entangled with the top

$$(\qquad\qquad\qquad)\otimes\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

!

$n=2$

(The above argument works for any n in which

$$\left(\sum \frac{1}{(\sqrt{2})^n}(-1)^{f(x_0 x_1\cdots x_{n-1})}|x_0\cdots x_{n-1}\rangle\right)\otimes\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

all possible $x_0, x_1 \cdots x_n$ combinations

Step 3 The Top Qubits Pass through $H^{\otimes 2}$ Gate

$$= \left((-1)^{f(0,0)} \quad \right.$$

$\sim \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \begin{pmatrix}0\\0\\0\\1\end{pmatrix} = \begin{pmatrix} (-1)^{f(0,1)} \\ (-1)^{f(1,0)} \\ (-1)^{f(1,1)} \end{pmatrix}$ (*)

$|00\rangle = \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$

$|01\rangle = \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$

$|10\rangle = \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}$

$|11\rangle = \begin{pmatrix}0\\1\end{pmatrix} \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$

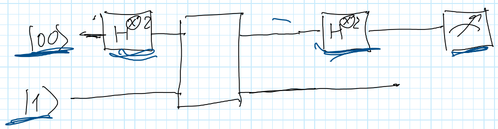
Passing this through Hadamard Gate $\otimes 2$

$$\boxed{H^{\otimes 2}} \begin{bmatrix} (-1)^{f(0,0)} \\ (-1)^{f(0,1)} \\ (-1)^{f(1,0)} \\ (-1)^{f(1,1)} \end{bmatrix} = \quad = \text{Eq. 4} \left(+ \right)$$

$|00\rangle \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$

- Lets check the Top entry

⟹ For The Deutsch – Jozsa Algorithm we consider



- This correspond to a top state in Eq(4) which results of the output

$$\begin{pmatrix} \frac{1}{4}\left((-1)^{f(0,0)} + (-1)^{f(0,1)} + (-1)^{f(1,0)} + (1)^{f(1,1)}\right) \\ \frac{1}{4}\left((-1)^{f(0,0)} - (-1)^{f(0,1)} + (-1)^{f(1,0)} - (1)^{f(1,1)}\right) \\ \frac{1}{4}\left((-1)^{f(0,0)} + (-1)^{f(0,1)} - (-1)^{f(1,0)} - (-1)^{f(1,1)}\right) \\ \frac{1}{4}\left((-1)^{f(0,0)} - (-1)^{f(0,1)} - (-1)^{f(1,0)} + (-1)^{f(1,1)}\right) \end{pmatrix}$$

- This is qubit which is being measured at the Top of output

Consider (*)

- if f_i – const function (a)

$f(0,0)=0$, $f(0,1)=0$, $f(1,0)=0$, $f(1,1)=0$ or (b) $f(0,0)=1$, $f(0,1)=1$, $f(1,0)=1$, $f(1,1)=1$

balanced

1	1
1	0
0	0
0	1

(a) $\frac{1}{4}\begin{pmatrix}1\\0\\0\\0\end{pmatrix} \leftarrow$ (b) $\frac{1}{4}\begin{pmatrix}1\\0\\0\\0\end{pmatrix} \leftarrow |00\rangle = \frac{1}{4}\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$

- if f_i – balanced $f(0,0)=0$ $f(0,1)=0$ $f(1,0)=1$ $f(1,1)=1$

$\frac{1}{4}\begin{pmatrix}0\\0\\1\\0\end{pmatrix}$

$\frac{1}{4}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$

$\frac{1}{4}\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$

Step 4. Measure of Top Qubits

- if functions are constant Top qubit gives $|00\rangle$

- Otherwise if balanced Top qubit gives $|01\rangle$, $|10\rangle$, $|11\rangle$

- Same is true if $F_{i=n}$ - functions are const

Top Qubit gives $|0000\cdots0_n\rangle$ - const.

If Balanced $|001_i\cdots0\rangle$ $\boxed{i=1\cdots n}$

- Deutsch - Jozsa algorithm needs only one measurement

instead of $2^{n-1}+1$.