HW-5

In[1]:= $$V = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{\sqrt{\rho^2+(z+d)^2}} - \frac{1}{\sqrt{\rho^2+\left(z+\frac{d}{2}\right)^2}} + \frac{2}{\sqrt{\rho^2+(z-d)^2}} - \frac{2}{\sqrt{\rho^2+\left(z-\frac{d}{2}\right)^2}}\right);$$

In[3]:= $$\text{Efield} = \text{FullSimplify}\left[-e_\rho\, D[V,\rho] - \frac{e_\phi}{\rho} D[V,\phi] - e_z\, D[V, z]\right]$$

Out[3]= $$-\frac{q\left(\frac{2(d-z)}{\left((d-z)^2+\rho^2\right)^{3/2}} - \frac{d+z}{\left((d+z)^2+\rho^2\right)^{3/2}} - \frac{8(d-2z)}{\left((d-2z)^2+4\rho^2\right)^{3/2}} + \frac{4(d+2z)}{\left((d+2z)^2+4\rho^2\right)^{3/2}}\right) e_z}{4\pi\epsilon_0} - \frac{q\left(-\frac{2\rho}{\left((d-z)^2+\rho^2\right)^{3/2}} + \frac{2\rho}{\left(\left(-\frac{d}{2}+z\right)^2+\rho^2\right)^{3/2}} + \frac{\rho}{\left(\left(\frac{d}{2}+z\right)^2+\rho^2\right)^{3/2}} - \frac{\rho}{\left((d+z)^2+\rho^2\right)^{3/2}}\right) e_\rho}{4\pi\epsilon_0}$$

In[5]:= $$E\rho = \text{Efield} \mathrel{/.} z \to 0 \mathrel{/.} e_0 \to e_z$$

Out[5]= $$-\frac{q\left(\frac{d}{(d^2+\rho^2)^{3/2}} - \frac{4d}{(d^2+4\rho^2)^{3/2}}\right) e_z}{4\pi\epsilon_0} - \frac{q\left(\frac{3\rho}{\left(\frac{d^2}{4}+\rho^2\right)^{3/2}} - \frac{3\rho}{(d^2+\rho^2)^{3/2}}\right) e_\rho}{4\pi\epsilon_0}$$

In[6]:= $$\text{Series}[E\rho, \{d, 0, 2\}]$$

Out[6]= $$-\frac{\left(q\sqrt{\rho^2}\, e_z\right) d}{8\left(\pi\rho^4\epsilon_0\right)} - \frac{27\left(q\sqrt{\rho^2}\, e_\rho\right) d^2}{32\left(\pi\rho^5\epsilon_0\right)} + O[d]^3$$

at $z=0$: dipole $\vec{E} = \frac{-qd}{8\pi\varepsilon_0}\frac{1}{\rho^3}\hat{e}_z$

quadrupole $\vec{E} = -\frac{27qd^2}{32\pi\varepsilon_0}\frac{1}{\rho^4}\hat{e}_\rho$

In[1]:= `f = r^2 Cos[θ] Sin[ϕ];`

In[3]:= `fcart = Simplify[f /. ϕ → ArcTan[y / x] /. θ → ArcCos[z/r] /. r → Sqrt[x^2 + y^2 + z^2]]`

Out[3]= $\dfrac{y\, z\, \sqrt{x^2+y^2+z^2}}{x \sqrt{1+\frac{y^2}{x^2}}}$

In[18]:= `fgrad = {D[fcart, x], D[fcart, y], D[fcart, z]};`

In[5]:= `gradsph = FullSimplify[fgrad /. x → r Sin[θ] Cos[ϕ] /. y → r Sin[θ] Sin[ϕ] /. z → r Cos[θ]]`

Out[5]= $\left\{-\dfrac{\sqrt{r^2}\, \mathrm{Cos}[\theta]^2\, \mathrm{Cot}[\theta]\, \mathrm{Sin}[\phi]}{\sqrt{\mathrm{Sec}[\phi]^2}},\ \sqrt{r^2}\, \mathrm{Cos}[\theta]\, \mathrm{Cos}[\phi]\, \sqrt{\mathrm{Sec}[\phi]^2}\, \left(\mathrm{Cos}[\theta]\, \mathrm{Cos}[\phi]^2\, \mathrm{Cot}[\theta] + \mathrm{Sin}[\theta]\right),\ \dfrac{\sqrt{r^2}\, (3+\mathrm{Cos}[2\theta])\, \mathrm{Tan}[\phi]}{2\sqrt{\mathrm{Sec}[\phi]^2}}\right\}$

In[6]:=
```
e_r = {Sin[θ] Cos[ϕ], Sin[θ] Sin[ϕ], Cos[θ]};
e_θ = {Cos[θ] Cos[ϕ], Cos[θ] Sin[ϕ], -Sin[θ]};
e_ϕ = {-Sin[ϕ], Cos[ϕ], 0};
```

In[12]:=
```
gradr = FullSimplify[Dot[gradsph, e_r]]
D[f, r]
```

Out[12]= $\dfrac{2\sqrt{r^2}\, \mathrm{Cos}[\theta]\, \mathrm{Tan}[\phi]}{\sqrt{\mathrm{Sec}[\phi]^2}}$ $= 2r\cos\theta\sin\phi$

Out[13]= $2\, r\, \mathrm{Cos}[\theta]\, \mathrm{Sin}[\phi]$

In[14]:=
```
gradθ = FullSimplify[Dot[gradsph, e_θ]]
(1/r) D[f, θ]
```

Out[14]= $-\dfrac{\sqrt{r^2}\, \mathrm{Sin}[\theta]\, \mathrm{Tan}[\phi]}{\sqrt{\mathrm{Sec}[\phi]^2}}$ $= -r\sin\theta\sin\phi$

Out[15]= $-r\, \mathrm{Sin}[\theta]\, \mathrm{Sin}[\phi]$

In[16]:=
```
gradϕ = FullSimplify[Dot[gradsph, e_ϕ]]
(1/(r Sin[θ])) D[f, ϕ]
```

Out[16]= $\dfrac{\sqrt{r^2}\, \mathrm{Cot}[\theta]}{\sqrt{\mathrm{Sec}[\phi]^2}}$ $= r\cot\theta\cos\phi$

Out[17]= $r\, \mathrm{Cos}[\phi]\, \mathrm{Cot}[\theta]$

EMM

HW-5

2-5 when $\vec{r}'$ fixed $\vec{r}$ rotating

$$\ddot{\vec{r}}' = \ddot{\vec{r}} + \dot{\vec{\omega}}\times\vec{r} + 2\vec{\omega}\times\dot{\vec{r}} + \vec{\omega}\times(\vec{\omega}\times\vec{r})$$

on earth $\ddot{\vec{r}}' = \vec{g}$ $\quad \dot{\vec{\omega}} = 0$ $\quad \vec{\omega}$ small.

so $\ddot{\vec{r}} = \vec{g} - 2(\vec{\omega}\times\dot{\vec{r}}) \Longrightarrow$ Coriolis acceleration

up to $\vec{\omega}$ $\quad O(\omega)$

$$\dot{\vec{r}} = \int \ddot{\vec{r}}\, dt = \vec{v} + \vec{g}t + O(\omega)$$

so up to $O(\omega)$

$$\ddot{\vec{r}} = \vec{g} - 2(\vec{\omega}\times(\vec{v}+\vec{g}t)) = \vec{g} - 2(\vec{\omega}\times\vec{v}) - 2(\vec{\omega}\times\vec{g})t$$

$$\dot{\vec{r}} = \vec{v} + \vec{g}t - 2(\vec{\omega}\times\vec{v})t - (\vec{\omega}\times\vec{g})t^2$$

$$\vec{r} = \vec{v}t + \frac{1}{2}\vec{g}t^2 - (\vec{\omega}\times\vec{v})t^2 - \frac{1}{3}(\vec{\omega}\times\vec{g})t^3$$

flight time $\tau = 2*(\text{time when } \vec{v}_t\cdot\vec{g} = 0)$

$$\vec{g}\cdot\dot{\vec{r}} = 0 = \vec{v}\cdot\vec{g} + g^2 t - 2(\vec{\omega}\times\vec{v})\cdot\vec{g}\,t - (\vec{\omega}\times\vec{g})\cdot\vec{g}\,t^2$$

$$(\vec{\omega}\times\vec{g})\cdot\vec{g} = (\vec{g}\times\vec{g})\cdot\vec{\omega} = 0$$

$$(\vec{\omega}\times\vec{v})\cdot\vec{g} = (\vec{g}\times\vec{\omega})\cdot v$$

so $\tau = -2\vec{v}\cdot\vec{g}/g^2 \dfrac{1}{1-2(\vec{g}\times\vec{\omega})\cdot\vec{v}/g^2}$

$$\tau = \tau_0\left(1 + 2(\vec{g}\times\vec{\omega})\cdot\vec{v}/g^2\right) + O(\omega^2)$$

$$\tau = \tau_0(1+\delta) \qquad \delta \sim O(\omega)$$

notice: $\vec{r}(\vec{\omega}=0) = \vec{v}\tau_0 + \frac{1}{2} g \tau_0^2$

so $\vec{r} = \vec{r}(\omega=0) + \vec{v}\tau_0\delta + \frac{1}{2}\vec{g}\, 2\tau_0^2\delta + \frac{1}{2}\vec{g}\tau_0^2\delta^2$

$\quad - (\vec{\omega}\times\vec{v})\tau_0^2(1+\delta)^2 - \frac{1}{3}(\vec{\omega}\times\vec{g})\tau_0^3(1+\delta)^3$

so up to $\mathcal{O}(\omega)$ only δ term remains

$$\vec{r} - \vec{r}(\omega=0) = \tau_0(\vec{v} + \vec{g}\tau_0)\delta - (\vec{\omega}\times\vec{v})\tau_0^2 - \frac{1}{3}(\vec{\omega}\times\vec{g})\tau_0^3$$

where $\tau_0 = -2\, \vec{v}\cdot\vec{g}/g^2$

$$\vec{r} - \vec{r}(\omega=0) = \frac{2\tau_0}{g^2}[(\vec{g}\times\vec{\omega})\cdot\vec{v}](\vec{g}\tau_0 + \vec{v}) - (\vec{\omega}\times\vec{v})\tau_0^2$$

$$- \frac{1}{3}(\vec{\omega}\times\vec{g})\tau_0^3 \qquad !!$$

for $\vec{v} \parallel \vec{\omega}$ $\quad v \sim 300\ m/s$ $\quad \tau_0 \sim 2\frac{1}{\sqrt{2}}\,300/9.8 \sim 43\,s$,

so range $\sim v\frac{1}{\sqrt{2}} * \tau_0 \sim 9000\,m \sim 9\,km$

$\vec{r} - \vec{r}(\omega=0) \sim 15\,m$! for $\omega = 7.3\times10^{-5}\ rad/s$

or: $\vec{g}\cdot\vec{r}(\text{landing}) = 0$,

so: $\vec{g}\cdot\vec{r} = \vec{v}\cdot\vec{g}\,t + \frac{1}{2}g^2 t^2 - (\vec{\omega}\times\vec{v})\cdot\vec{g}\, t^2 - \frac{1}{3}(\vec{\omega}\times\vec{g})\cdot\vec{g}\, t^3$

$0 = \vec{v}\cdot\vec{g} + \frac{1}{2}g^2\cancel{\tau} - (\vec{\omega}\times\vec{v})\cdot\vec{g}\,\tau \qquad = 0$

2.10

a) $\vec{\nabla}\times\vec{a}(\nabla\cdot\vec{a})+\vec{a}\times[\vec{\nabla}\times(\vec{\nabla}\times\vec{a})]+\vec{a}\times\nabla^2\vec{a}$

$=(\vec{\nabla}\cdot\vec{a})(\vec{\nabla}\times\vec{a})-\vec{a}\times\nabla(\nabla\cdot\vec{a})$

$+\vec{a}\times[\vec{\nabla}(\vec{\nabla}\cdot\vec{a})-\nabla^2\vec{a}]+\vec{a}\times\nabla^2\vec{a}$

$=(\vec{\nabla}\cdot\vec{a})(\vec{\nabla}\times\vec{a})$

b) $(\vec{\nabla}\times\vec{a})\cdot\boxed{(\vec{b}\times\vec{c})}$

$=((\vec{b}\times\vec{c})\times\vec{\nabla})\cdot\vec{a}$

$=[\vec{c}(\vec{b}\cdot\vec{\nabla})-\vec{b}(\vec{c}\cdot\vec{\nabla})]\cdot\vec{a}$

c) $\vec{a}\times(\vec{\nabla}\times\vec{a})=\vec{a}\cdot\vec{\nabla}\vec{a}-(\vec{a}\cdot\vec{\nabla})\vec{a}$

$=\vec{\nabla}(\frac{1}{2}a^2)-(\vec{a}\cdot\vec{\nabla})\vec{a}$

2-14 cylindrical $\hat{e}_\rho = \{\cos\phi, \sin\phi, 0\}$ in cartesian

$$\hat{e}_\phi = \{-\sin\phi, \cos\phi, 0\}$$

clearly $\frac{\partial}{\partial\phi}\hat{e}_\rho = \hat{e}_\phi \quad \frac{\partial}{\partial\phi}\hat{e}_\phi = -\hat{e}_\rho$

i) $\nabla^2 \vec{a} \qquad \vec{a} = \hat{e}_\rho$

$$\vec{\nabla}\cdot(\nabla\hat{e}_\rho) = \vec{\nabla}\cdot\left(\hat{e}_\rho \frac{\partial}{\partial\rho}\hat{e}_\rho + \hat{e}_\phi \frac{1}{\rho}\frac{\partial}{\partial\phi}(\hat{e}_\rho) + \hat{e}_z \frac{\partial}{\partial z}(\hat{e}_\rho)\right)$$

$$= (\vec{\nabla}\cdot)\left(0 + \underbrace{\hat{e}_\phi \frac{1}{\rho}(\hat{e}_\phi)} + 0\right)$$

$$= \frac{1}{\rho}\frac{\partial}{\partial\phi}\left(\frac{1}{\rho}\hat{e}_\phi\right) = -\frac{1}{\rho^2}\hat{e}_\rho$$

ii) $$\vec{\nabla}\times\hat{e}_\rho = \frac{1}{\rho}\begin{vmatrix} \hat{e}_\rho & \rho\hat{e}_\phi & \hat{e}_z \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z} \\ 1 & 0 & 0 \end{vmatrix} = 0$$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{a}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{a}) - \nabla^2\vec{a} = 0 \qquad \vec{a} = \hat{e}_\rho$$

so $\nabla(\vec{\nabla}\cdot\vec{a}) = \nabla^2\vec{a}$

iii) $\vec{\nabla}\cdot\hat{e}_\rho = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho\times 1) = \frac{1}{\rho}$

$$\nabla(\vec{\nabla}\cdot\hat{e}_\rho) = \nabla\frac{1}{\rho} = \hat{e}_\rho\frac{\partial}{\partial\rho}\left(\frac{1}{\rho}\right) + 0 + 0$$

$$\nabla(\vec{\nabla}\cdot\hat{e}_\rho) = -\frac{1}{\rho^2}\hat{e}_\rho$$

$$[\nabla(\vec{\nabla}\cdot\hat{e}_\rho)]\cdot\hat{e}_\rho = -\frac{1}{\rho^2}$$

clearly $\nabla^2(\hat{e}_\rho\cdot\hat{e}_\rho) = 0$

so $[\nabla(\vec{\nabla}\cdot\vec{a})]_\rho \neq \nabla^2 a_\rho = \nabla^2(\vec{a}\cdot\hat{e}_\rho)$

2-14 (b)

$$\nabla^2 \hat{e}_\rho = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\left(\frac{x}{\sqrt{x^2+y^2}}\hat{e}_x + \frac{y}{\sqrt{x^2+y^2}}\hat{e}_y + 0\right)$$

$$= \left[\frac{\partial^2}{\partial x^2}\left(\frac{x}{\sqrt{x^2+y^2}}\right) + \frac{\partial^2}{\partial y^2}\left(\frac{x}{\sqrt{x^2+y^2}}\right)\right]\hat{e}_x$$

$$\left[\frac{\partial^2}{\partial x^2}\frac{y}{\sqrt{x^2+y^2}} + \frac{\partial^2}{\partial y^2}\left(\frac{y}{\sqrt{x^2+y^2}}\right)\right]\hat{e}_y$$

$$\frac{\partial^2}{\partial x^2}\left(\frac{x}{(x^2+y^2)^{1/2}}\right) = \frac{\partial}{\partial x}\left[\frac{1}{(x^2+y^2)^{1/2}} - \frac{x^2}{(x^2+y^2)^{3/2}}\right]$$

$$= \frac{\partial}{\partial x}\frac{y^2}{(x^2+y^2)^{3/2}} = -3\frac{xy^2}{(x^2+y^2)^{5/2}}$$

$$\frac{\partial^2}{\partial y^2}\left(\frac{x}{\sqrt{x^2+y^2}}\right) = \frac{\partial}{\partial y}\left(-\frac{xy}{(x^2+y^2)^{3/2}}\right) = -\frac{x}{(x^2+y^2)^{3/2}} + 3\frac{xy^2}{(x^2+y^2)^{5/2}}$$

$$\text{so}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\frac{x}{\sqrt{x^2+y^2}}\right) = -\frac{x}{(x^2+y^2)^{3/2}}$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\frac{y}{\sqrt{x^2+y^2}}\right) = -\frac{y}{(x^2+y^2)^{3/2}}$$

$$\nabla^2 e_\rho = -\frac{x}{(x^2+y^2)^{3/2}}\hat{e}_x - \frac{x}{(x^2+y^2)^{3/2}}\hat{e}_y = -\frac{1}{\rho^2}\hat{e}_\rho$$

2.15 i) $\vec{\nabla}\cdot\vec{B} = 0$ ii) $\vec{\nabla}\cdot\vec{E} = 0$

iii) $\vec{\nabla}\times\vec{E} + \frac{\partial\vec{B}}{\partial t} = 0$ iv) $\vec{\nabla}\times\vec{B} - \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t} = 0$

let $\vec{B} = \vec{\nabla}\times\vec{A}$ $\vec{E} = -\vec{\nabla}\phi - \frac{\partial}{\partial t}\vec{A}$

$$\vec{\nabla}\cdot\vec{B} = \nabla\cdot(\vec{\nabla}\times\vec{A}) = 0 \quad ✓$$

$$\vec{\nabla}\times\vec{E} + \frac{\partial\vec{B}}{\partial t} = \cancel{-\vec{\nabla}\times\vec{\nabla}\phi} - \frac{\partial}{\partial t}\vec{\nabla}\times\vec{A} + \frac{\partial}{\partial t}\vec{\nabla}\times\vec{A} = 0 \quad ✓$$

$$\vec{\nabla}\times\vec{\nabla}\phi = 0$$

ii) $\vec{\nabla}\cdot\vec{E} = \nabla^2\phi + \frac{\partial}{\partial t}\vec{\nabla}\cdot\vec{A} = 0$

if $\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial\phi}{\partial t} = 0$

$$\frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A}) + \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = 0$$

$$\boxed{\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = 0}$$

iv) $\vec{\nabla}\times\vec{B} - \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t} = \vec{\nabla}\times(\vec{\nabla}\times\vec{A}) + \frac{1}{c^2}\frac{\partial}{\partial t}\vec{\nabla}\phi + \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = 0$

$$\nabla\times(\nabla\times\vec{A}) = \vec{\nabla}(\nabla\cdot\vec{A}) - \nabla^2\vec{A}$$

if $\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial\phi}{\partial t} = 0$ $\nabla(\nabla\cdot\vec{A}) + \frac{1}{c^2}\frac{\partial}{\partial t}\nabla\phi = 0$

$$\nabla(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A} + \frac{1}{c^2}\frac{\partial}{\partial t}\nabla\phi + \frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} = 0$$

$$\boxed{\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = 0}$$