HW-7

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = \alpha\omega_0^2\left(t + \gamma/\omega_0^2\right)$$

particular integral: $x = at + b$.

$$\gamma a + \omega_0^2 a t + \omega_0^2 b = \alpha\omega_0^2 t + \alpha\gamma$$

so: $a = \alpha$, $b = 0$

$\underline{x = \alpha t}$ is the particular solution.

Fourier series for $f(t) = t + \gamma/\omega_0^2$ (cosine)

$$b_0 = \int_0^1 \left(t + \gamma/\omega_0^2\right) dt = \frac{1}{2} + \gamma/\omega_0^2$$

$$b_n = 2\int_0^1 \left(t + \gamma/\omega_0^2\right)\cos n\pi t \, dt = 2\int_0^1 t\cos n\pi t\, dt$$

$$b_n = -\frac{2}{(n\pi)^2}\left(1 - (-1)^n\right)$$

equation becomes:

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = \alpha\omega_0^2\left[\frac{1}{2} + \gamma/\omega_0^2 + \sum \frac{-2}{(n\pi)^2}\left(1 - (-1)^n\right)\cos n\pi t\right.$$

consider general:

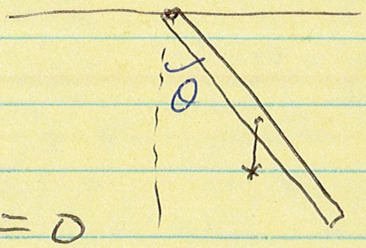
$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = P_n \cos n\pi t$$

$$x = A\cos(n\pi t + \phi)$$

$$A = \frac{P_n}{\sqrt{(\omega_0^2 - n^2\pi^2)^2 + \gamma^2 n^2\pi^2}}$$

$$\phi = \arctan\frac{-\gamma n\pi}{\omega_0^2 - n^2\pi^2}$$

2. a) $\tau = -\frac{1}{2}L \cdot Mg \cdot \sin\theta$



$$I = \frac{1}{3}ML^2$$

so $$\frac{1}{3}ML^2 \frac{d^2\theta}{dt^2} + \frac{1}{2}LMg\sin\theta = 0$$

$$\frac{d^2\theta}{dt^2} + \frac{3}{2}\frac{g}{L}\sin\theta = 0$$

$\theta << 1 \quad \sin\theta \sim \theta$

$$\ddot{\theta} + \frac{3}{2}\frac{g}{L}\theta = 0$$

$$\theta = \theta_0 \cos(\omega t) \qquad \omega = \sqrt{\frac{3g}{2L}}.$$

b) change variable $t \to t\sqrt{\frac{g}{L}}$

$$\frac{d^2\theta}{dt^2} + \frac{3}{2}\sin\theta = 0$$

MATHEMATICA DSolve with $\dot{\theta}(0) = 0 \quad \theta(0) = \frac{\pi}{2}$

then Solve t for $\theta = 0 \quad t = 1.51385$

$\omega = \frac{2\pi}{4t} = 1.03762$

(Compare to small angle oscillation $\omega = \sqrt{\frac{3}{2}} = 1.22474$

In[2]:= `DSolve[{θ''[t] + 3/2 Sin[θ[t]] == 0, θ[0] == π/2, θ'[0] == 0}, θ[t], t]`

$$\texttt{DSolve}\left[\left\{\theta''[t]+\frac{3}{2}\,\texttt{Sin}[\theta[t]]==0,\ \theta[0]==\frac{\pi}{2},\ \theta'[0]==0\right\},\ \theta[t],\ t\right]$$

Out[2]= $$\left\{\left\{\theta[t]\to 2\,\texttt{JacobiAmplitude}\left[\frac{1}{2}\left(-\sqrt{3}\,t+2\,\texttt{EllipticF}\left[\frac{\pi}{4},2\right]\right),2\right]\right\}\right\}$$

In[13]:= $$\texttt{N}\left[\texttt{Solve}\left[0==2\,\texttt{JacobiAmplitude}\left[\frac{1}{2}\left(-\sqrt{3}\,t+2\,\texttt{EllipticF}\left[\frac{\pi}{4},2\right]\right),2\right],t\right]\right]$$

Solve::ifun :

Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. >>

Out[13]= $$\{\{t\to 1.51385-1.72064\times 10^{-8}\,i\}\}$$

In[18]:=
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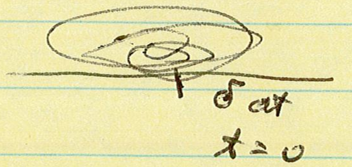
Period = 4 * t /. t → 1.5138456348012468`;
ω = 2 π / Period
```

Out[19]= 1.03762

In[21]:= $\texttt{N}\left[\texttt{Sqrt}\left[\frac{3}{2}\right]\right]$ $\longrightarrow$ small angle solution

Out[21]= 1.22474



3) $p = \rho y \frac{dy}{dt}$ $\qquad F = \rho y g$

$$F = \frac{dp}{dt}$$

$$\rho y g = \rho \left(\frac{dy}{dt}\right)^2 + \rho y \frac{d^2y}{dt}$$

or. $$\frac{d^2y}{dt^2} + \frac{1}{y}\left(\frac{dy}{dt}\right)^2 = g$$

let $\left(\frac{dy}{dt}\right)^2 = f(y)$

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{dy}{dt}\frac{d}{dy}\left(\frac{dy}{dt}\right)$$

$$= \frac{1}{2}\frac{d}{dy}\left(\frac{dy}{dt}\right)^2$$

so $$\frac{d^2y}{dt^2} = \frac{1}{2}\frac{d}{dy} f(y)$$

$$\frac{df}{dy} + \frac{2}{y} f = 2g$$

$\times y^2 \rightarrow$ $$y^2\frac{df}{dy} + 2yf = 2y^2 g$$

$$\frac{d}{dy}(y^2 f) = \frac{d}{dy}\left(\frac{2}{3}y^3 g\right)$$

$$f(y) = \frac{2}{3} g\left(y - \frac{\delta^3}{y^2}\right)$$

at $t \rightarrow \infty$ $\quad y \rightarrow \infty$ $\quad \frac{df(y)}{dy} = 2\frac{d^2y}{dt^2} = \frac{2}{3} g$

or $$\frac{d^2y}{dt^2} = \frac{1}{3} g$$

OR: from $y\frac{d^2y}{dt^2} + \left(\frac{dy}{dt}\right)^2 = gy$

time derivative: $\frac{dy}{dt}\frac{d^2y}{dt^2} + y\frac{d^3y}{dt^3} + 2\left(\frac{dy}{dt}\right)\frac{d^2y}{dt^2}$
$= g\frac{dy}{dt}$

at $t \to \infty$ $\frac{d^3y}{dt^3} = \frac{d}{dt}\left(\frac{d^2y}{dt^2}\right) \to 0$

$$3\left(\frac{dy}{dt}\right)\frac{d^2y}{dt^2} = g\left(\frac{dy}{dt}\right)$$

$$\frac{d^2y}{dt^2} = \frac{g}{3}$$

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