HW 8

(3) $$k \nabla^2 u = \frac{\partial u}{\partial t} \qquad u = R(r)\,T(t)$$

$$k\, T(t)\, \nabla^2 R(r) = R(r)\, T'(t)$$

$$k \frac{\nabla^2 R(r)}{R(r)} = \frac{T'(t)}{T(t)} = \alpha \text{ (constant)}$$

$$T'(t) = \alpha T(t) \Longrightarrow T(t) = C_1 e^{\alpha t}$$

in spherical coordinate:

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2}{\partial \phi^2}$$

if radially symmetric

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}$$

$$R''(r) + \frac{2}{r} R'(r) = \frac{\alpha}{k} R(r)$$

$$R''(r) + \frac{2}{r} R'(r) - \frac{\alpha}{k} R(r) = 0$$

let $R(r) = f(r) e^{\beta r}$

$$R'(r) = f'(r) e^{\beta r} + \beta f(r) e^{\beta r}$$

$$R''(r) = f''(r) e^{\beta r} + \beta f'(r) e^{\beta r} + \beta f'(r) e^{\beta r} + \beta^2 f(r) e^{\beta r}$$

$$f''(r) + 2\beta f'(r) + \beta^2 f(r) + \frac{2}{r} f'(r) + \frac{2\beta}{r} f(r) - \frac{\alpha}{k} f(r) = 0$$

$$f''(r) + 2\left(\beta + \frac{1}{r}\right) f'(r) + \left(\beta^2 + \frac{2\beta}{r} - \frac{\alpha}{k}\right) f(r) = 0$$

$f(r) = \frac{1}{r}$ satisfies equation

$$\beta^2 = \frac{\alpha}{k}$$

$$R(r) = \frac{C_1}{r} e^{\sqrt{\frac{\alpha}{k}}\, r} + \frac{C_2}{r} e^{-\sqrt{\frac{\alpha}{k}}\, r}$$

HW 8

1) $r = \dfrac{p}{1+e\cos\theta}$; $r^2 \dfrac{d\theta}{dt} = \sqrt{p}\sqrt{GM_s}$

so $\dfrac{d\theta}{dt} = \sqrt{GM_s}\,\dfrac{\sqrt{p}}{p^2}(1+e\cos\theta)^2$

since $a = \dfrac{p}{1-e^2} = 1\,AU$, $p = 0.75\,AU = \frac{3}{4}\,AU$

$e = 0.5$ $\quad b = \dfrac{p}{\sqrt{1-e^2}} = \dfrac{\sqrt{3}}{2}\,AU$

$\dfrac{d\theta}{dt} = \sqrt{GM_s}\sqrt{\frac{3}{4}}\,\dfrac{16}{9}(1+0.5\cos\theta)^2$

$GM_s = 6.674\times10^{-11} \times 1.989\times10^{30}\left(\dfrac{AU}{1.496\times10^{11}}\right)^3\left(\dfrac{24\times3600}{DAY}\right)^2$.

unit $[GM_s] = \dfrac{N m^2}{kg} = kg\, m/s^2\, \dfrac{m^2}{kg} = \dfrac{m^3}{s^2}$

$GM_s = 2.9597\times10^{-4}\,(AU)^3/(DAY)^2$,

so $\dfrac{d\theta}{dt} = 0.026487(1+0.5\cos\theta)^2$

$\theta_0 = 0 \quad \theta_1$ when $dt = 1$ (DAY)

$K_1 = 0.026487(1+0.5)^2 = 0.059596$

$K_2 = 0.026487(1+0.5\cos\frac{K_1}{2})^2 = 0.059578$

$K_3 = 0.026487(1+0.5\cos\frac{K_2}{2})^2 = 0.059578$

$K_4 = 0.026487(1+0.5\cos K_3)^2 = 0.059525$

$\theta_1 = \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4) = 0.059572$.

$\Delta A = \frac{1}{2}r^2(\theta_1 - \theta_0) = \frac{1}{2}\left(\frac{0.75}{1+0.5}\right)^2 \cdot \theta_1 = 7.4465\times10^{-3}$

$T = \dfrac{\pi a b}{\Delta A} = \dfrac{\pi \frac{\sqrt{3}}{2}}{\Delta A} = 365.36$ DAY!

$dt = 10$ DAY

$$K_1 = 0.026487(1+0.5)^2 * 10 = 0.59596$$

$$K_2 = 0.026487\left(1+0.5\cos\frac{K_1}{2}\right)^2 * 10 = 0.57858$$

$$K_3 = 0.026487\left(1+0.5\cos\frac{K_2}{2}\right)^2 * 10 = 0.57956$$

$$K_4 = 0.026487(1+0.5\cos K_3)^2 * 10 = 0.53284$$

$$\theta_{10} = \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4) = 0.57418$$

$$\Delta A = \frac{1}{2} r^2 (\theta_{10} - \theta_0) = \frac{1}{8}(\theta_{10}) = 7.177 \times 10^{-2}$$

$$T = \frac{\pi \frac{\sqrt{3}}{2}}{\Delta A} = 37.909 * 10 \text{ DAY}$$

$$r_0 = \frac{p}{1+e} = \frac{1}{2}$$

$$r_{10} = \frac{p}{1+e\cos\theta_{10}} = \frac{\frac{3}{4}}{1+0.5\cos\theta_{10}} = 0.5282$$

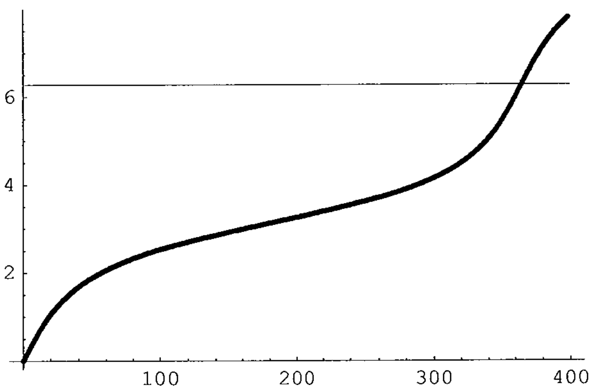
$$\Delta A = \frac{1}{2}(0.5141)^2 \cdot \theta_{10} = 0.07588$$

$$T = 35.85 * 10 \text{ DAY.}$$

```
fac = 0.026487;
step = 1;
i = 1; θ[i] = 0.; t[i] = 0.;

nstep = 400;
Do[{t0 = θ[i], k1 = step*fac*(1 + 0.5 Cos[t0])^2,
  k2 = step*fac*(1 + 0.5 Cos[t0 + k1/2])^2, k3 = step*fac*(1 + 0.5 Cos[t0 + k2/2])^2,
  k4 = step*fac*(1 + 0.5 Cos[t0 + k3])^2, i = i + 1, θ[i] = t0 + 1/6 (k1 + 2 k2 + 2 k3 + k4),
  t[i] = (i - 1)*step}, {i, nstep}];
```

```
In[19]:= orb = Table[{t[i], θ[i]}, {i, nstep}]; Show[Plot[2π, {i, 1, step*nstep}], ListPlot[orb]]
```

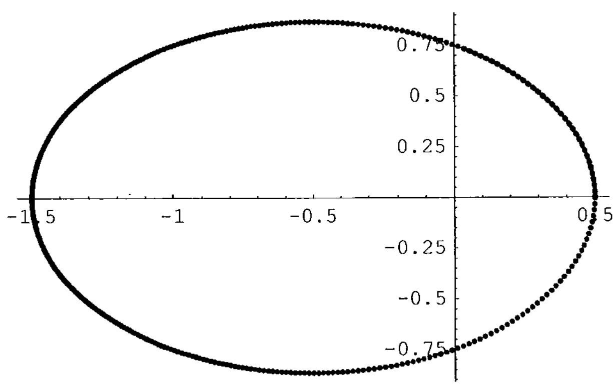


$\theta[366]/2\pi = 0.997908$

$\theta[367]/2\pi = 1.00739$

```
Out[19]= - Graphics -
```

```
In[18]:= ellipse = Table[{0.75/(1 + 0.5 Cos[θ[i]]) Cos[θ[i]], 0.75/(1 + 0.5 Cos[θ[i]]) Sin[θ[i]]}, {i, nstep}];
         ListPlot[ellipse]
```

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Out[18]= - Graphics -
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```
In[20]:= NIntegrate[1/(fac (1 + 0.5 Cos[t])^2), {t, 0, 2π}]

Out[20]= 365.221
```

$\int dt = \int_0^{2\pi} \frac{d\theta}{fac(1+e\cos\theta)^2}$

$T = 365.22$ DAY