Frobenius method: Theorem 6.3, case 3: $r_1 = r_2$.

Solve the D.E on $0<x<R$, $(x_0=0)$

$$xy'' + y' + 2y = 0$$

We seek $y(x) = \sum_{n=0}^{\infty} c_n (x)^{n+r}$. so $y'(x) = \sum_{n=0}^{\infty} (n+r) c_n x^{n+r-1}$,

$$y''(x) = \sum_{n=0}^{\infty} (n+r)(n+r-1) c_n x^{n+r-2}, \quad (c_0 \neq 0)$$

Reporting those in D.E, we find:

$$\sum_{n=0}^{\infty} (n+r)(n+r-1) c_n x^{n+r-1} + \sum_{n=0}^{\infty} (n+r) c_n x^{n+r-1} + 2\sum_{n=0}^{\infty} c_n x^{n+r} = 0, \text{ for all } 0<x<R$$

Regrouping the first two sums, we get

$$\sum_{n=0}^{\infty} (n+r)(n+r-1+1) c_n x^{n+r-1} + 2\sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

Replace $n-1$ with n

$$\sum_{n=-1}^{\infty} (n+1+r)^2 c_{n+1} x^{n+r} + 2\sum_{n=0}^{\infty} c_n x^{n+r} = 0, \text{ for all } 0<x<R$$

Take out the first term from the left sum, and regroup the remaining sums to get:

$$r^2 c_0 x^{-1+r} + \sum_{n=0}^{\infty} \left[(n+1+r)^2 c_{n+1} + 2c_n\right] x^{n+r} = 0, \text{ for all } 0<x<R$$

Therefore the indicial equation is $c_0 r^2 = 0$ or $r^2 = 0$, as $c_0 \neq 0$

So $r_1 = 0 = r_2$. Set $r=0$, the recursion formula is then

$$(n+1)^2 c_{n+1} + 2c_n = 0, \quad n = 0, 1, 2, 3, \ldots$$

or $$c_{n+1} = -\frac{2}{(n+1)^2} c_n, \quad n = 0, 1, 2, 3, \ldots$$

Thus
$$c_n = -\frac{2}{n^2} c_{n-1}, \quad n = 1, 2, 3, \ldots$$
$$c_{n-1} = -\frac{2}{(n-1)^2} c_{n-2}, \quad n = 2, 3, \ldots$$
$$\vdots$$
$$c_1 = -\frac{2}{1^2} c_0$$

Multiplying those equalities side by side and cancelling common terms on both sides:

$$\cancel{c_1 c_2 \cdots c_{n-1}} \cdot c_n = \frac{(-2)^n}{1^2(2^2)(\cdots)n^2} c_0 \cancel{c_1 c_2 \cdots c_{n-1}}$$

Consequently, $$c_n = \frac{(-2)^n c_0}{(1 \cdot 2 \cdots n)^2} = \frac{(-2)^n}{(n!)^2} c_0$$

Hence, setting ... we find:

$$y_1(x) = \sum_{n=0}^{\infty} \frac{(-2)^n}{(n!)^2} x^n, \quad \text{(Remember that } 0! = 1\text{, by convention)}$$

We shall now find a linearly independent solution y_2.
Given that $r_1 = r_2 = 0$, we know that y_2 has the form

$$y_2(x) = x \sum_{n=0}^{\infty} c_n^* x^n + y_1(x)\ln(x), \quad \text{with } c_0^* \neq 0$$

We shall use the reduction of order method to get y_2.
Seek $y_2(x) = y_1(x)v(x)$, for some unknown function v to be determined.

$$y_2' = y_1'v + y_1 v', \quad y_2'' = y_1''v + 2y_1'v' + y_1 v''$$

Reporting in D.E, we obtain:

$$x y_2'' + y_2' + 2y_2 = x y_1''v + 2x y_1' v' + x y_1 v'' + y_1' v + y_1 v' + 2 y_1 v = 0$$

or

$$\underbrace{(x y_1'' + y_1' + 2y_1)}_{0 \text{ as } y_1 \text{ solves D.E.}} v + x y_1 v'' + (2x y_1' + y_1) v' = 0$$

So

$$x y_1 v'' + (2x y_1' + y_1) v' = 0$$

Set $w = v'$, the latter D.E becomes

$$x y_1 w' + (2x y_1' + y_1) w = 0$$

or $w' + \left(\frac{2y_1'}{y_1} + \frac{1}{x}\right) w = 0$, first-order linear D.E

Hence

$$w(x) = c e^{-\int \left(\frac{2y_1'(x)}{y_1(x)} + \frac{1}{x}\right)dx} \quad c = \text{arbitrary constant}$$

$$= c e^{-2\ln|y_1(x)| - \ln(x)}$$

$$= c e^{-\ln(y_1(x)^2 x)} = c e^{\ln\left(\frac{1}{x y_1(x)^2}\right)}$$

$$= \frac{c}{x(y_1(x))^2} = \frac{1}{x(y_1(x))^2}$$

Now

$$y_1(x) = 1 - 2x + x^2 - \frac{2}{9}x^3 + \frac{1}{36}x^4 + \cdots$$

$$y_1(x)^2 = \left(1 - 2x + x^2 - \frac{2}{9}x^3 + \frac{1}{36}x^4 + \cdots\right)\left(1 - 2x + x^2 - \frac{2}{9}x^3 + \frac{1}{36}x^4 + \cdots\right)$$

$$= 1 - 4x + 6x^2 - \frac{40}{9}x^3 + \frac{35}{18}x^4 + \cdots$$

$$\frac{1}{xy_1^2} = \frac{1}{x}\cdot\frac{1}{y_1^2}$$

Note: $\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 \ldots$ provided $|x|<1$

$$= \frac{1}{x}\cdot\frac{1}{1-\left(4x - 6x^2 + \frac{40}{9}x^3 - \frac{35}{18}x^4 + \cdots\right)}$$

$$= \frac{1}{x}\left(1 + \left(4x - 6x^2 + \frac{40}{9}x^3 - \frac{35}{18}x^4\right) + 16x^2 - 48x^3 + \frac{644}{9}x^4 + 64x^3 + \cdots\right.$$

$$= \frac{1}{x}\left(1 + 4x + 10x^2 + \frac{184}{9}x^3 + \cdots\right)$$

$$= \frac{1}{x} + 4 + 10x + \frac{184}{9}x^2 + \cdots$$

$$= w(x)$$

$$= v'(x)$$

So $v(x) = \ln(x) + 4x + 5x^2 + \frac{184}{27}x^3 + \cdots$

$$y_2(x) = y_1(x)\ln(x) + y_1(x)\left(4x + 5x^2 + \frac{184}{27}x^3 + \cdots\right)$$

$$= y_1(x)\ln(x) + \left(1 - 2x + x^2 - \frac{2}{9}x^3 + \cdots\right)\left(4x + 5x^2 + \frac{184}{27}x^3 + \cdots\right)$$

$$= y_1(x)\ln(x) + 4x - 3x^2 + \frac{22}{7}x^3 + \cdots$$

Hence

$$y_2(x) = x\left(4 - 3x + \frac{22}{7}x^2 + \cdots\right) + y_1(x)\ln(x),$$ as claimed.

The general solution of the D.E is

$y(x) = c_1 y_1(x) + c_2 y_2(x)$, where c_1 and c_2 are arbitrary constants.