Assignment 1 – key

1. a) $A_m = \begin{pmatrix} 1 & m+1 & 0 & 2m \\ m & 0 & 1 & 1 \\ 2m+1 & 1 & m+1 & 1 \\ 0 & 0 & m+1 & m+1 \end{pmatrix}$

b) A_m is singular if $\det(A_m) = 0$.

$$\det(A_m) = (m+1)\begin{vmatrix} 1 & m+1 & 0 & 2m \\ m & 0 & 1 & 1 \\ 2m+1 & 1 & m+1 & 1 \\ 0 & 0 & 1 & 1 \end{vmatrix}$$

Expand along last row

$$= (m+1)\left\{ \begin{vmatrix} 1 & m+1 & 0 \\ m & 0 & 1 \\ 2m+1 & 1 & m+1 \end{vmatrix} - \begin{vmatrix} 1 & m+1 & 2m \\ m & 0 & 1 \\ 2m+1 & 1 & 1 \end{vmatrix} \right\}$$

Expand along 1^{st} row

Expand along 2^{nd} row

$$= (m+1)\{1(0-1) - (m+1)(m(m+1) - (2m+1))$$
$$\quad - [-m(m+1-2m) - (1-(m+1)(2m+1))]\}$$
$$= (m+1)\{-1 - (m+1)(m^2-m-1) - [+m^2 - m - \cancel{1} + 2m^2 + 3m + \cancel{1}]\}$$
$$= (m+1)\{-\cancel{1} - (m+1)(m^2-m-1) - (m^2-m-1) - 2m^2 - 3m - 1\}$$
$$= (m+1)\{-2 - (m+1)(m^2-m-1) - 3m^2 - 2m + 1\}$$
$$= (m+1)\{-(m+1)(m^2-m-1) - 3m^2 - 2m - 1\}$$
$$= (m+1)\{-m(m^2-m-1) - m^2 + m + \cancel{1} - 3m^2 - 2m - \cancel{1}\}$$
$$= m(m+1)\{-m^2 + \cancel{m} + \cancel{1} - \cancel{m} + \cancel{1} - 3m - \cancel{2}$$
$$= m(m+1)(-m^2 - 3m)$$
$$= -m^2(m+1)(m+3)$$

$\det(A_m) = 0$ for $m = 0$, $m = -1$, or $m = -3$.

$m = 0$. Linear system becomes in augmented matrix form

$$\left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & a \\ 0 & 0 & 1 & 1 & b \\ 1 & 1 & 1 & 1 & c \\ 0 & 0 & 1 & 1 & d \end{array}\right) \xrightarrow{-r_1+r_3} \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & a \\ 0 & 0 & 1 & 1 & b \\ 0 & 0 & 1 & 1 & c-a \\ 0 & 0 & 1 & 1 & d \end{array}\right)$$

$$\xrightarrow[-r_2+r_4]{-r_2+r_3} \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & a \\ 0 & 0 & 1 & 1 & b \\ 0 & 0 & 0 & 0 & c-a-b \\ 0 & 0 & 0 & 0 & d-b \end{array}\right)$$

System is consistent if and only if $d = b$ and $c = a+b$. In this case we get $x_1 + x_2 = a \rightarrow x_2 = -x_1 + a$, $x_3 + x_4 = b \rightarrow x_4 = -x_3 + b$

$S = \{ (\alpha, -\alpha + a, \beta, -\beta + b)^T ; \alpha, \beta \in \mathbb{R} \}$

$m = -1$.

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & -2 & a \\ -1 & 0 & 1 & 1 & b \\ -1 & 1 & 0 & 1 & c \\ 0 & 0 & 0 & 0 & d \end{array}\right) \xrightarrow[r_1+r_3]{r_1+r_2} \left(\begin{array}{cccc|c} 1 & 0 & 0 & -2 & a \\ 0 & 0 & 1 & -1 & b+a \\ 0 & 1 & 0 & -1 & c+a \\ 0 & 0 & 0 & 0 & d \end{array}\right)$$

$$\xrightarrow{r_2 \leftrightarrow r_3} \left(\begin{array}{cccc|c} 1 & 0 & 0 & -2 & a \\ 0 & 1 & 0 & -1 & c+a \\ 0 & 0 & 1 & -1 & b+a \\ 0 & 0 & 0 & 0 & d \end{array}\right)$$

System is consistent if and only if $d = 0$.

$x_3 - x_4 = b + a \rightarrow x_3 = x_4 + b + a$

$x_2 - x_4 = c + a \rightarrow x_2 = x_4 + c + a$

$x_1 - 2x_4 = a \rightarrow x_1 = 2x_4 + a$

$S = \{ (2\alpha + a, \alpha + a + c, \alpha + a + b, \alpha)^T ; \alpha \in \mathbb{R} \}$

$m = -3.$

$$\left(\begin{array}{cccc|c} 1 & -2 & 0 & -6 & a \\ -3 & 0 & 1 & 1 & b \\ -5 & 1 & -2 & 1 & c \\ 0 & 0 & -2 & -2 & d \end{array}\right) \xrightarrow[3r_1+r_2]{5r_1+r_3}$$

$$\left(\begin{array}{cccc|c} 1 & -2 & 0 & -6 & a \\ 0 & -6 & 1 & -17 & b+3a \\ 0 & -9 & -2 & -29 & c+5a \\ 0 & 0 & -2 & -2 & d \end{array}\right) \xrightarrow[r_4/(-2)]{-\frac{3}{2}r_2+r_3}$$

$$\left(\begin{array}{cccc|c} 1 & -2 & 0 & -6 & a \\ 0 & -6 & 1 & -17 & b+3a \\ 0 & 0 & -\frac{7}{2} & -\frac{7}{2} & c+5a-\frac{3}{2}(b+3a) \\ 0 & 0 & 1 & 1 & -d/2 \end{array}\right) \xrightarrow{r_3 \leftrightarrows r_4}$$

$$\left(\begin{array}{cccc|c} 1 & -2 & 0 & -6 & a \\ 0 & -6 & 1 & -17 & b+3a \\ 0 & 0 & 1 & 1 & -\frac{d}{2} \\ 0 & 0 & -\frac{7}{2} & -\frac{7}{2} & c+\frac{a}{2}-\frac{3b}{2} \end{array}\right) \xrightarrow{\frac{7}{2}r_3+r_4}$$

$$\left(\begin{array}{cccc|c} 1 & -2 & 0 & -6 & a \\ 0 & -6 & 1 & -17 & b+3a \\ 0 & 0 & 1 & 1 & -\frac{7d}{4} \\ 0 & 0 & 0 & 0 & -\frac{7}{4}d+c+\frac{a}{2}-\frac{3b}{2} \end{array}\right)$$

System is consistent if and only if $2a-6b+4c-7d=0$

$x_3 = -x_4 - \frac{7d}{4}, \quad x_2 = \frac{1}{6}x_3 - \frac{17}{6}x_4 = -\frac{18}{6}x_4 - \frac{7d}{24} = -3x_4 - \frac{7d}{24}$

$x_1 = 2x_2 + 6x_4 = -6x_4 - \frac{7d}{12} + 6x_4 = -\frac{7d}{12}$

$S = \{(-\frac{7d}{12}, -3\alpha - \frac{7d}{24}, -\alpha - \frac{7d}{4}, \alpha)^T ; \alpha \in \mathbb{R}\}$

2. $I_n + A - 5A^2 + 7A^5 - 9A^{11} = O_{M_n} \Rightarrow I_n = -A + 5A^2 - 7A^5 + 9A^{11}$

So $I_n = A(-I_n + 5A - 7A^4 + 9A^{10})$

Hence A is nonsingular and $A^{-1} = -I_n + 5A - 7A^4 + 9A^{10}$.

3. $A \xrightarrow{2r_1+r_3} B,\quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{2r_1+r_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} = E$

$EA = B$

$A \xrightarrow{-2c_3+c_1} C,\quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{-2c_3+c_1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix} = F$

$AF = C$.

4. $A \neq O_{M_n}$, $B \neq O_{M_n}$, $C \neq O_{M_n}$, and $ABC = O_{M_n}$. Show at least two of those matrices are singular.

Proceed by contradiction. Suppose that at most ~~one~~ of the matrices is singular. Then ~~exactly~~ one matrix is singular, and two matrices are nonsingular; say B and C are nonsingular. Then

$$ABC(BC)^{-1} = O_{M_n}(BC)^{-1} = O_{M_n}$$

$$ABCC^{-1}B^{-1} = O_{M_n} \to ABB^{-1} = O_{M_n} \to A = O_{M_n},$$ which is a contradiction, as $A \neq O_{M_n}$ by hypothesis. Hence at least two of those matrices are singular.