Assignment 2 - key

1. $u_1 = (1, m, 3)^T$, $u_2 = (2, -1, 4)^T$, $u_3 = (m, -2, 1)^T$.
u_1, u_2, u_3 are linearly independent if $\det(u_1, u_2, u_3) \neq 0$. Now

$$\begin{vmatrix} 1 & 2 & m \\ m & -1 & -2 \\ 3 & 4 & 1 \end{vmatrix} \overset{-mr_1+r_2}{\underset{-3r_1+r_3}{=}} \begin{vmatrix} 1 & 2 & m \\ 0 & -1-2m & -2-m^2 \\ 0 & -2 & 1-3m \end{vmatrix} = -(1+2m)(1-3m) + 2(-2-m^2)$$
$$= -1 + 3m - 2m + 6m^2 - 4 - 2m^2$$
$$= 4m^2 + m - 5$$
$$= (m-1)(4m+5)$$

$(4m+5)(m-1) = 0 \longrightarrow m = -5/4$ or $m = 1$

So u_1, u_2, u_3 are linearly independent for $m \neq -5/4$ and $m \neq 1$.

2. $v_1 = (1, 2, -1, m-2)^T$, $v_2 = (2, 1, m+1, -4)^T$, $v_3 = (-3, m-1, 2, 2)^T$, $v_4 = (-2, 1, -3, 4)^T$
$\mathbb{R}^4 = \operatorname{Span}(v_1, v_2, v_3, v_4)$ iff v_1, v_2, v_3, v_4 are linearly independent, by Theorem 3.4.3.

$$\det(v_1, v_2, v_3, v_4) = \begin{vmatrix} 1 & 2 & -3 & -2 \\ 2 & 1 & m-1 & 1 \\ -1 & m+1 & 2 & -3 \\ m-2 & -4 & 2 & 4 \end{vmatrix} \overset{-2r_1+r_2}{\underset{\substack{r_1+r_3 \\ (2-m)r_1+r_4}}{=}}$$

$$= \begin{vmatrix} 1 & 2 & -3 & -2 \\ 0 & -3 & m+5 & 5 \\ 0 & m+3 & -1 & -5 \\ 0 & -2m & 3m-4 & 2m \end{vmatrix} \overset{\frac{(m+3)}{3}r_2+r_3}{\underset{-\frac{2m}{3}r_2+r_4}{=}}$$

$$= \begin{vmatrix} 1 & 2 & -3 & -2 \\ 0 & -3 & m+5 & 5 \\ 0 & 0 & -1+\frac{(m+5)(m+3)}{3} & -5+\frac{(m+3)5}{3} \\ 0 & 0 & 3m-4-\frac{2m}{3}(m+5) & 2m-\frac{10m}{3} \end{vmatrix}$$

$$= -3\left[\left(2m - \frac{10m}{3}\right)\left(-1 + \frac{(m+5)(m+3)}{3}\right) - \left(-5 + \frac{5(m+3)}{3}\right)\left(3m - 4 - \frac{2m(m+5)}{3}\right)\right]$$

$$= -3\left[-\frac{4m}{3}\left(\frac{-3+m^2+8m+15}{3}\right) - \frac{5m}{3}\left(\frac{9m-12-2m^2-10m}{3}\right)\right]$$

$$= \frac{m}{3}\left[4(m^2+8m+12) + 5(-2m^2-m-12)\right]$$

$$= \frac{m}{3}\left[-6m^2+27m-12\right]$$

$$= -m\left[2m^2-9m+4\right]$$

$$= -m(2m-1)(m-4)$$

$$= 0 \longrightarrow m = 0,\ m = 1/2 \text{ or } m = 4.$$

Hence v_1, v_2, v_3, v_4 span $\mathbb{R}^4$ iff $m \neq 0$, and $m \neq 1/2$ and $m \neq 4$.

3) $u_1 = (1,3,1)^T$, $u_2 = (-1,2,1)^T$, $u_3 = (3,2,-2)^T$

$v_1 = (-1,-1,3)^T$, $v_2 = (1,1,4)^T$, $v_3 = (-4,1,3)^T$

a) $$\det(u_1,u_2,u_3) = \begin{vmatrix} 1 & -1 & 3 \\ 3 & 2 & 2 \\ 1 & 1 & -2 \end{vmatrix} = 1(-4-2)+1(-6-2)+3(3-2) = -6-8+3 \neq 0$$

Therefore u_1, u_2, u_3 are linearly independent; so they span $\mathbb{R}^3$ by Theorem 3.4.3; hence $\{u_1,u_2,u_3\}$ is a basis of $\mathbb{R}^3$.

$$\det(v_1,v_2,v_3) = \begin{vmatrix} -1 & 1 & -4 \\ -1 & 1 & 1 \\ 3 & 4 & 3 \end{vmatrix} = -(3-4)-(-3-3)-4(-4-3) = 1+6+28 = 35 \neq 0;$$ so, as above

$\{v_1, v_2, v_3\}$ is a basis of $\mathbb{R}^3$ too.

b) Set $B = [u_1,u_2,u_3]$, $D = [v_1,v_2,v_3]$. We shall find $T_{D\to B}$.

Set $$V = \begin{pmatrix} -1 & 1 & -4 \\ -1 & 1 & 1 \\ 3 & 4 & 3 \end{pmatrix},\quad U = \begin{pmatrix} 1 & -1 & 3 \\ 3 & 2 & 2 \\ 1 & 1 & -2 \end{pmatrix}.$$ Then $T_{D\to B} = U^{-1}V$

We start with the augmented matrix $(U|V)$ and find its RREF.

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 3 & -1 & 1 & -4 \\ 3 & 2 & 2 & -1 & 1 & 1 \\ 1 & 1 & -2 & 3 & 4 & 3 \end{array}\right) \xrightarrow[-r_1+r_2]{-3r_1+r_2} \left(\begin{array}{ccc|ccc} 1 & -1 & 3 & -1 & 1 & -4 \\ 0 & 5 & -7 & 2 & -2 & 13 \\ 0 & 2 & -5 & 4 & 3 & 7 \end{array}\right)$$

$$\xrightarrow{-\frac{2r_2}{5}+r_3} \left(\begin{array}{ccc|ccc} 1 & -1 & 3 & -1 & 1 & -4 \\ 0 & 5 & -7 & 2 & -2 & 13 \\ 0 & 0 & -\frac{11}{5} & \frac{16}{5} & \frac{19}{5} & \frac{9}{5} \end{array}\right) \xrightarrow[r_2/5]{-\frac{5}{11}r_3} \left(\begin{array}{ccc|ccc} 1 & -1 & 3 & -1 & 1 & -4 \\ 0 & 1 & -\frac{7}{5} & \frac{2}{5} & -\frac{2}{5} & \frac{13}{5} \\ 0 & 0 & 1 & -\frac{16}{11} & -\frac{19}{11} & -\frac{9}{11} \end{array}\right)$$

$$\xrightarrow[\frac{7}{5}r_3+r_2]{r_2+r_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 8/5 & -\frac{3}{5} & \frac{3}{5} & -\frac{7}{5} \\ 0 & 1 & 0 & -\frac{18}{11} & -\frac{31}{11} & \frac{16}{11} \\ 0 & 0 & 1 & -\frac{16}{11} & -\frac{19}{11} & -\frac{9}{11} \end{array}\right) \xrightarrow{-\frac{8}{5}r_3+r_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{19}{11} & \frac{37}{11} & -\frac{1}{11} \\ 0 & 1 & 0 & -\frac{18}{11} & -\frac{31}{11} & \frac{16}{11} \\ 0 & 0 & 1 & -\frac{16}{11} & -\frac{19}{11} & -\frac{9}{11} \end{array}\right)$$

Hence $$T_{D\to B} = \begin{pmatrix} \frac{19}{11} & \frac{37}{11} & -\frac{1}{11} \\ -\frac{18}{11} & -\frac{31}{11} & \frac{16}{11} \\ -\frac{16}{11} & -\frac{19}{11} & -\frac{9}{11} \end{pmatrix}$$

c) $v = 3v_1 - 2v_2 + v_3$

$$[v]_D = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}, \quad [v]_B = T_{D \to B}[v]_D = \begin{pmatrix} \frac{19}{11} & \frac{37}{11} & -\frac{1}{11} \\ -\frac{18}{11} & -\frac{31}{11} & \frac{16}{11} \\ -\frac{16}{11} & -\frac{19}{11} & -\frac{9}{11} \end{pmatrix} \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$$

$$[v]_B = \begin{pmatrix} -18/11 \\ 24/11 \\ -19/11 \end{pmatrix}$$

4. $A \in M_3$, $A^3 = O_{M_3}$, $A^2 \neq O_{M_3}$. Let $x \in \mathbb{R}^3$ with $A^2x \neq O_{\mathbb{R}^3}$.

Show that x, Ax and A^2x are linearly independent.

Let α, β, γ be scalars such that $\alpha x + \beta Ax + \gamma A^2 x = O_{\mathbb{R}^3}$. to show that $\alpha = \beta = \gamma = 0$. Apply A^2 to both sides of equation to get

$$A^2(\alpha x + \beta Ax + \gamma A^2 x) = O_{\mathbb{R}^3}. \text{ So,}$$

$$\alpha A^2 x + \beta A^3 x + \gamma A^4 x = O_{\mathbb{R}^3}. \text{ Now } A^3 = O_{M_3}; \text{ so } A^4 = O_{M_3}.$$

Hence $\alpha A^2 x = O_{\mathbb{R}^3}$, and we know that $A^2 x \neq O_{\mathbb{R}^3}$. Therefore $\alpha = 0$.

We now have

$$\beta Ax + \gamma A^2 x = O_{\mathbb{R}^3}$$

Apply A to both sides to get

$$\beta A^2 x + \underbrace{\gamma A^3 x}_{O_{\mathbb{R}^3}} = O_{\mathbb{R}^3};$$ so that $\beta A^2 x = O_{\mathbb{R}^3}$, and $\beta = 0$ since $A^2 x \neq O_{\mathbb{R}^3}$. Hence $\gamma A^2 x = O_{\mathbb{R}^3}$; so $\gamma = 0$ since $A^2 x \neq O_{\mathbb{R}^3}$.

Therefore x, Ax and $A^2 x$ are linearly independent vectors in $\mathbb{R}^3$.

By Theorem 3.4.3, we derive that $\mathbb{R}^3 = \text{Span}(x, Ax, A^2x)$.

Hence x, Ax, A^2x form a basis of $\mathbb{R}^3$, by the definition of a basis.