Problem session 1 — key

A

1. system is inconsistent

2. $$\left(\begin{array}{cccc|c} 0 & 1 & 1 & 1 & 0 \\ 3 & 0 & 3 & -4 & 7 \\ 1 & 1 & 1 & 2 & 6 \\ 2 & 3 & 1 & 3 & 6 \end{array}\right) \xrightarrow{r_1 \leftrightarrows r_3} \left(\begin{array}{cccc|c} 1 & 1 & 1 & 2 & 6 \\ 3 & 0 & 3 & -4 & 7 \\ 0 & 1 & 1 & 1 & 0 \\ 2 & 3 & 1 & 3 & 6 \end{array}\right)$$

$$\xrightarrow[-2r_1+r_4]{-3r_1+r_2} \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 6 \\ 0 & -3 & 0 & -10 & -11 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & -1 & -6 \end{array}\right) \xrightarrow[-r_3+r_4]{r_2/3} \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 6 \\ 0 & -1 & 0 & -\frac{10}{3} & -\frac{11}{3} \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & -2 & -2 & -6 \end{array}\right) \xrightarrow{r_2+r_3}$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 6 \\ 0 & -1 & 0 & -10/3 & -11/3 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & -2 & -2 & -6 \end{array}\right) \xrightarrow{2r_3+r_4} \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 6 \\ 0 & -1 & 0 & -10/3 & -11/3 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -6 \end{array}\right)$$

System is inconsistent: last row reads $0 = -6$.

B.

$$\left(\begin{array}{ccc|c} 2 & -1 & 3 & 0 \\ -4 & m & 1 & 0 \\ -2 & 1 & m & 0 \\ 10 & -5 & -6 & 0 \end{array}\right) \xrightarrow[\substack{r_1+r_3 \\ -5r_1+r_4}]{2r_1+r_2} \left(\begin{array}{ccc|c} 2 & -1 & 3 & 0 \\ 0 & m-2 & 7 & 0 \\ 0 & 0 & m+3 & 0 \\ 0 & 0 & -21 & 0 \end{array}\right)$$

system reduces to: (working backward):

$-21x_3 = 0 \longrightarrow x_3 = 0$

$(m+3)(0) = 0$ ✓

$(m-2)x_2 + 7(0) = 0$, so $(m-2)x_2 = 0$; if $m \neq 2$, then $x_2 = 0$, and

$2x_1 - (0) + 3(0) = 0 \longrightarrow x_1 = 0$; So if $m \neq 2$, system has the trivial solution only. If $m = 2$, we find $2x_1 - x_2 = 0$; the solution of the system is the line with equations $2x_1 = x_2$, $x_3 = 0$,

$S = \{\alpha(1,2,0)^T;\ \alpha \in \mathbb{R}\}$

C.

$$A_0 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \xrightarrow{-r_1+r_3} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \xrightarrow[-r_2+r_4]{-r_2+r_3}$$

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = RREF$$

$$A_{-1} = \begin{pmatrix} 1 & 0 & 0 & -2 \\ -1 & 0 & 1 & 1 \\ -1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow[r_1+r_3]{r_1+r_2} \begin{pmatrix} 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{r_2 \leftrightarrows r_3}$$

$$\begin{pmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} = RREF$$

$$A_1 = \begin{pmatrix} 1 & 2 & 0 & 2 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 2 & 1 \\ 0 & 0 & 2 & 2 \end{pmatrix} \xrightarrow[-3r_1+r_3]{-r_1+r_2} \begin{pmatrix} 1 & 2 & 0 & 2 \\ 0 & -2 & 1 & -1 \\ 0 & -5 & 2 & -5 \\ 0 & 0 & 2 & 2 \end{pmatrix} \xrightarrow[r_4/2]{-\frac{5}{2}r_2+r_3}$$

$$\begin{pmatrix} 1 & 2 & 0 & 2 \\ 0 & -2 & 1 & -1 \\ 0 & 0 & -\frac{1}{2} & -\frac{5}{2} \\ 0 & 0 & 1 & 1 \end{pmatrix} \xrightarrow[\frac{1}{2}r_4+r_3]{-r_4+r_2} \begin{pmatrix} 1 & 2 & 0 & 2 \\ 0 & -2 & 0 & -2 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{pmatrix} \xrightarrow[r_3 \rightleftarrows r_4]{r_2+r_1}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & -2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -2 \end{pmatrix} \xrightarrow[-r_4/2]{-r_2/2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow[-r_4+r_2]{-r_4+r_3} I_4 = RREF$$

D. At node A: $i_1 - i_2 + i_3 = 0$
— " — " B: $i_1 - i_2 + i_4 = 0$
— " — " C: $i_3 - i_5 + i_6 = 0$
— " — " D: $i_4 - i_5 + i_6 = 0$

At upper loop: $4i_1 + 2i_2 = 8$ or $2i_1 + i_2 = 4$
At middle loop: $2i_2 + 4i_5 = 0$ or $i_2 + 2i_5 = 0$
At lower loop: $4i_5 + 5i_6 = 10$

E. 1) $(A^TA)^T = A^T(A^T)^T = A^TA$; so A^TA is symmetric.
$(AA^T)^T = (A^T)^TA^T = AA^T$; so AA^T is symmetric.

2) Since A and A^T have the same diagonal entries, if $A = (a_{ij})$, and A is skew-symmetric, then
$a_{ii} = -a_{ii}$, for all $i = 1, 2, \ldots, n$
So $2a_{ii} = 0$; and so $a_{ii} = 0$ for all i; hence A has only zeros on its diagonal.

3) Suppose $AB = BA$. Show $(AB)^T = AB$.
$(AB)^T = B^TA^T = BA = AB$, so AB is symmetric if $AB = BA$.
Suppose $(AB)^T = AB$. Show $AB = BA$.
$BA = B^TA^T$ as A & B are symmetric.
$= (AB)^T$, by the transpose property
$= AB$, as $(AB)^T = AB$.

4) $AA^{-1} = I_n \Rightarrow (AA^{-1})^T = I_n^T = I_n$
$\Rightarrow (A^{-1})^TA^T = I_n$
$\Rightarrow A^T$ is nonsingular, and $(A^T)^{-1} = (A^{-1})^T$

5) $m = 1$: $(A^1)^{-1} = A^{-1}$. I.H: Let $m \geq 1$. Suppose that A^m is nonsingular and $(A^m)^{-1} = (A^{-1})^m$. Show A^{m+1} is nonsingular and $(A^{m+1})^{-1} = (A^{-1})^{m+1}$.
$A^{m+1} = A^mA$ is nonsingular as a product of nonsingular matrices.
$(A^{m+1})^{-1} = A^{-1}(A^m)^{-1} = A^{-1}(A^{-1})^m$, by I.H.
$= (A^{-1})^{m+1}$.

6) One shows by induction that for any $p \geq 2$, $I_n - A^p = (I_n - A)(I_n + A + \cdots + A^{p-1})$ and
Now if A is nilpotent of order p, then $A^p = 0$, and
$I_n = (I_n - A)(I_n + A + \cdots + A^{p-1})$; so $I_n - A$ is invertible, and
$(I_n - A)^{-1} = I_n + A + \cdots + A^{p-1}$.