# Problem Set 3 - MAS 3105 - SUM 15

1. Define on $\mathbb{R}^2$ a mapping $N$ by
$$N((u,v)) = \int_0^1 |u + tv|\,dt, \quad (u,v) \text{ in } \mathbb{R}^2.$$
Show that $N$ is a norm on $\mathbb{R}^2$, and find $N((1,-2))$.

2. On $C^1([0,1])$, define the mappings
$$N_1(f) = |f(0)| + \int_0^1 |f'(t)|\,dt, \quad f' = \text{derivative of } f$$
$$N_2(f) = \int_0^1 |f'(t)|\,dt$$
a) Show that $N_1$ is a norm on $C^1([0,1])$.
b) Is $N_2$ a norm on $C^1([0,1])$?

3. Let $E$ be an inner vector space. Let $U$, $V$ be subspaces of $E$. Show that a) $(U+V)^\perp = U^\perp \cap V^\perp$, b) $(U \cap V)^\perp = U^\perp + V^\perp$.

4. On $M_3$, consider the inner product
$$\langle A, B \rangle = \sum_{i=1}^{3} \sum_{j=1}^{3} a_{ij} b_{ij}.$$
Let $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$ $\quad B = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 1 & 1 & 2 \end{pmatrix}$

a) Find the projection $P_B(A)$ of $A$ onto $B$, and $A - P_B(A)$.
b) Find the distance between $A$ and $B$.

5. Let $S = \text{Span}((1,0,-2,1)^T, (0,1,3,-2)^T)$. Find a basis for $S^\perp$.

6. If $A \in M_{m,n}$ of rank $r$, what are the dimensions of $N(A)$ and $N(A^T)$?

7. Let $A \in M_{m,n}$. Show that
a) if $x \in N(A^TA)$, then $Ax \in R(A) \cap N(A^T)$.
b) $N(A^TA) = N(A)$.
c) $A$ and $A^TA$ have the same rank.
d) if $A$ has linearly independent columns, then $A^TA$ is nonsingular.