If E is a type III elementary matrix, then E is invertible.

Proof. Suppose that E is obtained from I_n by multiplying, say, the first row of I_n by a nonzero α, and adding the resulting row to the i^{th} row of I_n; so

$$E = \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ \alpha & 0 & \cdots\ 1 & & 0 \\ 0 & & & \ddots & \vdots \\ \vdots & & 0 & 1 & 0 \\ 0 & & & & 1 \end{pmatrix} \begin{array}{l} \\ \\ \\ \text{— } i^{th} \text{ row } (i \neq 1) \\ \\ \\ \\ \end{array}$$

Kronecker symbol

We shall find a matrix $D \in M_n$ such that
$ED = I_n = DE$.

Set $E = (e_{jk})$, $D = (d_{jk})$, $I_n = (\delta_{jk})$, $\delta_{jk} = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}$

We shall find d_{jk} for each j and k

By the definition of the product of two matrices, we have ($ED = I_n$):

$$\delta_{jk} = \sum_{r=1}^{n} e_{jr} d_{rk}$$

Let $j \neq i$, then $e_{jr} = \begin{cases} 1 & \text{if } r = j \\ 0 & \text{if } r \neq j \end{cases}$

So $\delta_{jk} = e_{jj} d_{jk}$, the sum reduces to a single term

So $d_{jk} = \delta_{jk}$ for all j, k with $j \neq i$

If $j = i$, then $e_{i1} = \alpha$, $e_{ii} = 1$, $e_{ir} = 0$ for $r \neq 1$ and $r \neq i$

so $\delta_{ik} = e_{i1} d_{1k} + e_{ii} d_{ik} = \alpha d_{1k} + d_{ik}$

If $k = 1$, then $\delta_{i1} = 0$ since $i \neq 1$; so $0 = \alpha + d_{i1}$, and $d_{i1} = -\alpha$

If $k = i$, then $\delta_{ii} = 1 = \alpha(0) + d_{ii} = 1$

If $k \neq 1$ and $k \neq i$, then $\delta_{ik} = 0 = \alpha(0) + d_{ik}$; so $d_{ik} = 0$.

Hence E is invertible and (You can check $DE = I_n$)

$$E^{-1} = D = \begin{array}{r} \\ \\ \\ \\ \\ i^{th}\text{ row} \longrightarrow \\ \\ \\ \\ \\ \end{array} \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 & \cdots & 0 \\ 0 & 1 & & & & & \\ \vdots & & \ddots & & & & \\ \vdots & & & & & 0 & \\ 0 & & & & & & \\ -2 & \cdots & \cdots & 1 & & & \\ 0 & & & & \ddots & & \\ \vdots & & & & & \ddots & \\ \vdots & & 0 & & & & 0 \\ 0 & & & & & & 1 \end{pmatrix}$$

Example If $n = 3$ and

$$E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{pmatrix}, \text{ then } E^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{pmatrix}$$