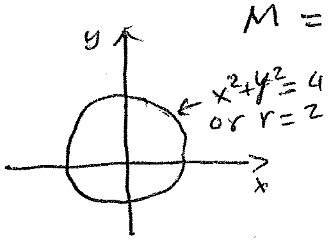
MAC 2313 (Multivariable Calculus) – Answers

QUIZ 11, Friday November 18, 2016

Name: **PID:**

Remember that no documents or calculators, or any other electronic devices are allowed during the quiz. Also remember that you won't get any credit(s) if you do not show the steps to your answers.

1. [5] a) Find the mass M of the solid bounded by the paraboloid $z = 4 - x^2 - y^2$ and the xy-plane, assuming that its density is $\delta(x, y, z) = x^2 + y^2 + 2z$.



$$M = \iiint_G (x^2+y^2+2z)\,dV = \int_0^{2\pi}\int_0^2\int_0^{4-r^2} r(r^2+2z)\,dz\,dr\,d\theta$$

$$= 2\pi \int_0^2 r\,(r^2 z + z^2]_0^{4-r^2}\,dr$$

$$= 2\pi \int_0^2 r\big(r^2(4-r^2)+(4-r^2)^2\big)dr = 2\pi \int_0^2 4r^3 - r^5 + (16-8r^2+r^4)r\,dr$$

$$= 2\pi \int_0^2 16r - 4r^3\,dr = 2\pi\left[8r^2 - r^4\right]_0^2$$

$$= 2\pi\,(32-16) = 32\pi$$

b) Write down the triple integrals for finding the coordinates $(\bar{x}, \bar{y}, \bar{z})$ of its center of gravity, include all integration limits, but do not evaluate any of the integrals involved.

$$\bar{x} = \frac{1}{M}\int_0^{2\pi}\int_0^2\int_0^{4-r^2} r\cos\theta \cdot r(r^2+2z)\,dz\,dr\,d\theta$$

$$\bar{y} = \frac{1}{M}\int_0^{2\pi}\int_0^2\int_0^{4-r^2} r\sin\theta \cdot r(r^2+2z)\,dz\,dr\,d\theta$$

$$\bar{z} = \frac{1}{M}\int_0^{2\pi}\int_0^2\int_0^{4-r^2} z\, r(r^2+2z)\,dz\,dr\,d\theta$$

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2. [5] a) Find the Jacobian $\frac{\partial(x,y)}{\partial(u,v)}$ of the transformation $u = \frac{y}{x}$, $v = xy$, and express it in terms of u and v.

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} -\frac{y}{x^2} & \frac{1}{x} \\ y & x \end{vmatrix} = -\frac{y}{x} - \frac{y}{x} = -\frac{2y}{x} = -2u$$

Hence $\frac{\partial(x,y)}{\partial(u,v)} = -\frac{1}{2u}$

b) Use an appropriate change of variables to evaluate the integral $I = \iint_R \cos(xy)\,dA$, if R is the region in the first quadrant enclosed by the lines $y = x$, $y = 2x$, $xy = \frac{\pi}{2}$, $xy = \pi$.

$\Downarrow$ $\frac{y}{x} = 1$ $\quad$ $\Downarrow$ $\frac{y}{x} = 2$

Set $u = \frac{y}{x}$, $v = xy$

$$I = \int_1^2\int_{\pi/2}^{\pi} \cos(v)\left|\frac{-1}{2u}\right| dv\,du = \int_1^2 \frac{1}{2u}\,du \cdot \left[\sin v\right]_{\pi/2}^{\pi}$$

$$= \frac{1}{2}\ln u\Big]_1^2 \,[0-1]$$

$$= \frac{1}{2}(\ln 2 - 0)(-1) = -\frac{1}{2}\ln 2$$