MAP 2302 (Differential Equations) — Answers
QUIZ 9, Friday April 6, 2018

Name: **PID:**

We have: $\mathcal{L}(t^n)(s) = n!/s^{n+1}$, $\mathcal{L}(\sin(bt))(s) = b/(s^2+b^2)$, $\mathcal{L}(\cos(bt))(s) = s/(s^2+b^2)$), $\mathcal{L}(e^{at})(s) = 1/(s-a)$.

1. [6] Use Laplace transform to solve the initial-value problem:
$y'' - 5y' + 6y = 7 - 4u_3(t)$, $y(0) = 0$, $y'(0) = 0$. (Show all your work)

Set $Y(s) = \mathcal{L}y(s)$

$$\mathcal{L}(y'' - 5y' + 6y)(s) = \mathcal{L}(7 - 4u_3)(s) = \frac{7}{s} - \frac{4e^{-3s}}{s}$$

$$s^2 Y(s) - sy(0) - y'(0) - 5(sY(s) - y(0)) + 6Y(s) = \frac{7}{s} - \frac{4e^{-3s}}{s}$$

$$(s^2 - 5s + 6)Y(s) = \frac{7}{s} - \frac{4e^{-3s}}{s}; \quad Y(s) = \frac{7}{s(s-2)(s-3)} - \frac{4e^{-3s}}{s(s-2)(s-3)}$$

$$\frac{1}{s(s-2)(s-3)} = \frac{A}{s} + \frac{B}{s-2} + \frac{C}{s-3} = \frac{A(s-2)(s-3) + Bs(s-3) + Cs(s-2)}{s(s-2)(s-3)}$$

So $A(s-2)(s-3) + Bs(s-3) + Cs(s-2) = 1$, for all $s \neq 0, 2, 3$

For A, set $s = 0$: $6A = 1 \to A = 1/6$, For B, set $s = 2$: $-2B = 1 \to B = -1/2$

For C, set $s = 3$: $3C = 1 \to C = 1/3$

$$y(t) = \mathcal{L}^{-1}Y(t) = 7\left(\frac{1}{6} - \frac{e^{2t}}{2} + \frac{e^{3t}}{3}\right) - 4u_3(t)\left(\frac{1}{6} - \frac{e^{2(t-3)}}{2} + \frac{e^{3(t-3)}}{3}\right)$$

$$= \begin{cases} 7\left(\frac{1}{6} - \frac{e^{2t}}{2} + \frac{e^{3t}}{3}\right), & 0 < t < 3 \\ \frac{1}{2} - \frac{7}{2}e^{2t} + 2e^{2(t-3)} - \frac{4}{3}e^{3(t-3)}, & t > 3 \end{cases}$$

2. [2] Find the inverse Laplace transform: (Show all your work)

$$\mathcal{L}^{-1}\left(\frac{2se^{-\pi s}}{(s^2+9)} - \frac{3}{s^2+16}\right)(t) = \mathcal{L}^{-1}\left(\frac{2se^{-\pi s}}{s^2+9}\right)(t) - \frac{3}{4}\mathcal{L}^{-1}\left(\frac{4}{s^2+16}\right)(t)$$

$$= 2u_\pi(t)\cdot\cos 3(t-\pi) - \frac{3}{4}\sin(4t)$$

$$= -2u_\pi(t)\cos(3t) - \frac{3}{4}\sin(4t)$$

3. [2] Let $f(t) = \begin{cases} t, & 0 < t < \pi, \\ -3, & t > \pi. \end{cases}$

$$f(t) = t(1 - u_\pi(t)) - 3u_\pi(t)$$
$$= t - (t-\pi)u_\pi(t) - (3+\pi)u_\pi(t)$$

Find the Laplace transform: (Show all your work)

$$\mathcal{L}(f)(s) = \mathcal{L}(t)(s) - \mathcal{L}((t-\pi)u_\pi(t))(s) - (3+\pi)\mathcal{L}(u_\pi)(s)$$

$$= \frac{1}{s^2} - \frac{e^{-\pi s}}{s^2} - (3+\pi)\frac{e^{-\pi s}}{s}$$