Simultaneous controllability and stabilization of some uncoupled wave and plate equations

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AIMS Conference
Orlando

July 4, 2012

Overview

- Simultaneous controllability of wave equations.

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- Simultaneous stabilization of wave equations.

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- Simultaneous controllability of plate equations.
- Simultaneous stabilization of plate equations.
- Some extensions and open problems.

Notations

Ω= bounded domain in $\mathbb{R}^N$, $N \geq 1$,
Γ= boundary of is smooth,
$T > 0$, $Q = \times (0, T)$
$\omega =$ nonvoid open subset in Ω.
$a_1,\ a_2, ...,\ a_q$, $(q \geq 2)$ are pairwise distinct positive constants.

Controllability

Consider the controllability problem: Given $(y_j^0, y_j^1)_j \in \left(L^2(\Omega) \times H^{-1}(\Omega)\right)^q$, find a control $v \in [H^1(0, T; L^2(\omega))]'$ such that if the q-tuple $(y_j)_j$ solves the system

$$
\begin{aligned}
& y_{jtt} - a_j \Delta y_j = v 1_\omega \text{ in } Q \\
& y_j = 0 \text{ on } \Gamma \times (0, T) \\
& y_j(x, 0) = y_j^0(x), \quad y_{jt}(x, 0) = y_j^1(x) \text{ in } \Omega, \\
& j = 1,\ 2, ...,\ q,
\end{aligned}
$$

then for each $j = 1,\ 2, ...,\ q$

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y_j(x, T) = 0, \quad y_{jt}(x, T) = 0, \text{ in } \Omega.
$$

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- Lions' HUM reduces exact controllability to an inverse (observability) estimate for the adjoint system.

Observability

Consider the uncoupled adjoint system

$$\begin{array}{l} u_{jtt} - a_j \Delta u_j = 0 \text{ in } Q \\ u_j = 0 \text{ on } \Gamma \times (0, T) \\ u_j(x,0) = u_j^0(x), \quad u_{jt}(x,0) = u_j^1(x) \text{ in } \Omega, \quad j = 1,\ 2, ...,\ q, \end{array}$$

where $(u_j^0, u_j^1) \in H_0^1(\Omega) \times L^2(\Omega)$ for each j.

Haraux (1988) showed for arbitrary ω:

- If $\sum_{j=1}^{q} u_j(x,t) = 0$ in $\omega \times (0,T)$ then $u_j^0 = 0, \quad u_j^1 = 0$ in $\Omega, \quad \forall j$.

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- If $N = 1$ and T is large enough, or $\omega = \Omega$, then for all j and all $(u_j^0, u_j^1) \in L^2(\Omega) \times H^{-1}(\Omega)$

$$\sum_{j=1}^{q} \{||u_j^0||^2_{L^2(\Omega)} + ||u_j^1||^2_{H^{-1}(\Omega)}\} \leq C \int_0^T \int_\omega |\sum_{j=1}^{q} u_j(x,t)|^2 \, dxdt.$$

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- **(GCC)** [Bardos-Lebeau-Rauch, 1988, 1992]: every ray of geometric optics enters ω in a time less than T.

Theorem 1

Let T_0 denote the best controllability time for a single wave equation with unit speed of propagation. Suppose that

$T > T_0 \max\{a_j^{-\frac{1}{2}}; j = 1,\ 2,\ ...,\ q\}$ and (ω, T) satisfies (GCC). There exists a constant $C > 0$ such that for all $(u_j^0, u_j^1) \in H_0^1(\Omega) \times L^2(\Omega)$, $j = 1,\ 2, ...,\ q$:

$$\sum_{j=1}^{q} \{||u_j^0||^2_{H_0^1(\Omega)} + ||u_j^1||^2_{L^2(\Omega)}\} \leq C \int_0^T \int_\omega |\sum_{j=1}^{q} u_{jt}(x,t)|^2 \, dxdt,$$

with $C = C(\Omega, \omega, T, (a_j)_j, q)$.

Proof: key elements

- Thanks to Bardos-Lebeau-Rauch

$$\sum_{j=1}^{q}\{||u_j^0||^2_{H_0^1(\Omega)}+||u_j^1||^2_{L^2(\Omega)}\} \leq C\int_0^T r(t)\int_\omega \eta(x)\sum_{j=1}^{q}|u_{jt}(x,t)|^2\,dxdt.$$

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- Elementary algebra shows

$$\sum_{j=1}^{q}\{||u_j^0||^2_{H_0^1(\Omega)}+||u_j^1||^2_{L^2(\Omega)}\} \leq C\int_0^T\int_\omega|\sum_{j=1}^{q}u_{jt}(x,t)|^2\,dxdt \\ -2C\sum_{1\leq j<k\leq q}\int_Q r\eta u_{jt}u_{kt}\,dxdt.$$

- With $a_j \neq a_k$ for $j \neq k$, a combination of algebra and calculus shows

$$\sum_{j=1}^{q}\{||u_j^0||^2_{H_0^1(\Omega)} + ||u_j^1||^2_{L^2(\Omega)}\}$$

$$\leq C \int_0^T \int_\omega |\sum_{j=1}^{q} u_{jt}(x,t)|^2 \, dxdt$$

$$+C \int_Q \sum_{j=1}^{q} |u_j(x,t)|^2 \, dxdt.$$

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$$+C\int_Q\sum_{j=1}^{q}|u_j(x,t)|^2\,dxdt.$$

- Claim:

$$\int_Q\sum_{j=1}^{q}|u_j(x,t)|^2\,dxdt \leq C_0\int_0^T\int_\omega\left|\sum_{j=1}^{q} u_{jt}(x,t)\right|^2\,dxdt,$$

$$\forall (u_j^0, u_j^1)_j \in (H_0^1(\Omega)\times L^2(\Omega))^q.$$

Suppose that the claim fails. Then there are initial data in $(H_0^1(\Omega) \times L^2(\Omega))^q$ for which

$$\int_Q \sum_{j=1}^{q} |u_j(x,t)|^2\, dxdt = 1, \quad \sum_{j=1}^{q} u_{jt}(x,t) = 0 \text{ in } \omega \times (0,T).$$

The contradiction follows from

Unique continuation result.

Lemma

Let ω be an arbitrary nonvoid open subset in Ω. Let T, the constants a_js, and the initial data be given as in Theorem 1. Then

$$\sum_{j=1}^{q} u_{jt}(x,t) = 0 \text{ in } \omega \times (0,T) \Rightarrow u_j \equiv 0 \text{ in } Q.$$

Remark

It follows from Theorem 1 that for all $(u_j^0, u_j^1)_j \in \left(L^2(\Omega) \times H^{-1}(\Omega)\right)^q$

$$\widehat{E}(0) \leq C_0 \int_0^T \int_\omega \left| \sum_{j=1}^q u_j(x,t) \right|^2 \, dxdt,$$

where $2\widehat{E}(0) = \sum_{j=1}^q \left(||u_j^0||^2_{L^2(\Omega)} + ||u_j^1||^2_{H^{-1}(\Omega)} \right)$.

Stabilization

Given $(y_j^0, y_j^1)_j \in \left(H_0^1(\Omega) \times L^2(\Omega)\right)^q$, and a function $d \in L^\infty(\Omega)$, $d \geq 0$, consider the damped system

$$\begin{array}{l} y_{jtt} - a_j \Delta y_j + d \sum_{k=1}^q y_{kt} = 0 \text{ in } \Omega \times (0, \infty) \\ y_j = 0 \text{ on } \Gamma \times (0, \infty) \\ y_j(x,0) = y_j^0(x), \quad y_{jt}(x,0) = y_j^1(x), \text{ in } \Omega, \\ j = 1, \ 2, ..., \ q. \end{array}$$

The total energy is given, for all $t \geq 0$, by

$$2E(t) = \sum_{j=1}^{q} \int_\Omega \{|y_{jt}(x,t)|^2 + a_j |\nabla y_j(x,t)|^2\} \, dx,$$

and it is a nonincreasing function of the time variable as

$$\frac{dE}{dt} = -\int_\Omega d(x) \left| \sum_{k=1}^{q} y_{kt}(x,t) \right|^2 dx.$$

Let ω be a nonvoid open subset in Ω, and suppose that the damping is effective in ω, viz.: $\exists a_0 > 0 : d(x) \geq a_0$ a.e. in ω. The two questions that we would like to answer are the following:

- does the energy $E(t)$ decays to zero as $t \to \infty$?

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- does the energy $E(t)$ decays to zero as $t \to \infty$?
- If ω satisfies (GCC), do we have a uniform exponential decay of $E(t)$ in the energy space?

Theorem 2

Let $(y_j^0, y_j^1)_j \in \left(H_0^1(\Omega) \times L^2(\Omega)\right)^q$. Suppose that the constants a_j, $j = 1,\ 2,\ ...,\ q$, are pairwise distinct.

i) Assume that ω is a nonvoid open subset in Ω, and that the damping is effective in ω. Then the energy E satisfies $\lim_{t\to\infty} E(t) = 0$.

ii) Assume that ω satisfies (GCC), and suppose that the damping is effective in ω. There exists positive constants $M = M(\Omega, \omega, T, a, q, d)$, and $\mu = \mu(\Omega, \omega, T, a, q, d)$ such that the following energy decay estimate holds

$$E(t) \le Me^{-\mu t}E(0),\ \text{for all } t \ge 0.$$

Sketch of the proof of Theorem 2

1. If one denotes by A the underlying unbounded operator, then A generates a C_0 semigroup of contractions $(S(t))_{t\geq 0}$ on $H = (H_0^1(\Omega) \times L^2(\Omega))^q$. Further, A has a compact resolvent; so the spectrum $\sigma(A)$ is discrete. Next, one shows that A has no purely imaginary eigenvalue. The stability theorem in Arendt-Batty (1988) yields the claimed strong stability result.

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2. Thanks to Theorem 1 above, and a result of Haraux (1989), which establishes an equivalence between observability and stabilization for second order evolution equations with bounded damping operators, the claimed exponential decay follows.

Controllability

Consider the controllability problem: Given $(z_j^0, z_j^1)_j \in \left(L^2(\Omega) \times H^{-2}(\Omega)\right)^q$, find a control $v \in [H^1(0, T; L^2(\omega))]'$ such that if the q-tuple $(z_j)_j$ solves the system

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&z_j(x, 0) = z_j^0(x), \quad z_{jt}(x, 0) = z_j^1(x) \text{ in } \Omega, \\
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\end{aligned}$$

then for each $j = 1,\ 2, ...,\ q$

$$z_j(x, T) = 0, \quad z_{jt}(x, T) = 0, \text{ in } \Omega.$$

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Inverse inequality

Consider the uncoupled adjoint system

$$\begin{array}{l} w_{jtt} + a_j \Delta^2 w_j = 0 \text{ in } Q \\ w_j = 0, \quad \partial_\nu w_j = 0 \text{ on } \Gamma \times (0, T) \\ w_j(x,0) = w_j^0(x), \quad w_{jt}(x,0) = w_j^1(x) \text{ in } \Omega, \\ j = 1,\ 2, ...,\ q, \end{array}$$

where $(w_j^0, w_j^1) \in H_0^1(\Omega) \times L^2(\Omega)$ for each j.

Theorem 3

Let $T > 0$ be arbitrary. Suppose that ω is big enough (cf. e.g. [Russell (1973), Lions (1988)]). Assume that the constants a_j, $j = 1,\ 2,\ ...,\ q$, are pairwise distinct. There exists a constant $C > 0$ such that for all $(w_j^0, w_j^1) \in H_0^2(\Omega) \times L^2(\Omega)$, $j = 1,\ 2, ...,\ q$:

$$\sum_{j=1}^{q} \{||w_j^0||^2_{H_0^2(\Omega)} + ||w_j^1||^2_{L^2(\Omega)}\} \leq C \int_0^T \int_\omega \left| \sum_{j=1}^{q} w_{jt}(x,t) \right|^2 dxdt,$$

with $C = C(\Omega, \omega, T, (a_j)_j, q)$.

Remark

It follows from Theorem 3 that for all $(w_j^0, w_j^1)_j \in \left(L^2(\Omega) \times H^{-2}(\Omega)\right)^q$

$$\widetilde{E}(0) \leq C_0 \int_0^T \int_\omega \left| \sum_{j=1}^q w_j(x,t) \right|^2 \, dxdt,$$

where $2\widetilde{E}(0) = \sum_{j=1}^q \left(||w_j^0||^2_{L^2(\Omega)} + ||w_j^1||^2_{H^{-2}(\Omega)} \right)$.

Stabilization

Given $(z_j^0, z_j^1)_j \in \left(H_0^2(\Omega) \times L^2(\Omega)\right)^q$, and a function $d \in L^\infty(\Omega)$, $d \geq 0$, consider the damped system

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z_j(x,0) = z_j^0(x), \quad z_{jt}(x,0) = z_j^1(x), \text{ in } \Omega, \\
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The total energy is now given, for all $t \geq 0$, by

$$2E(t) = \sum_{j=1}^{q} \int_\Omega \{|z_{jt}(x,t)|^2 + a_j |\Delta z_j(x,t)|^2\}\, dx,$$

and it is a nonincreasing function of the time variable as

$$\frac{dE}{dt} = -\int_\Omega d(x) \left| \sum_{k=1}^{q} z_{kt}(x,t) \right|^2 dx$$

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Theorem 4

Let $(z_j^0, z_j^1)_j \in \left(H_0^2(\Omega) \times L^2(\Omega)\right)^q$. Suppose that the constants a_j, $j = 1,\ 2,\ ...,\ q$, are pairwise distinct.

i) Let ω be a nonvoid open subset in Ω, and suppose that the damping is effective in ω. Then the energy E satisfies $\lim_{t\to\infty} E(t) = 0$.

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- The case of nonconstant coefficients may be discussed using Riemannian geometry (cf. Lasiecka-Triggiani-Yao (1999)).

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- A potential of the form $a_j p(x)$ may be added to the wave or plate equations discussed. The potential p being nonnegative in the stabilization problems.
- The case of nonconstant coefficients may be discussed using Riemannian geometry (cf. Lasiecka-Triggiani-Yao (1999)).
- The case of boundary controllability or stabilization is widely open in higher space dimensions. For 1-d boundary controllability, cf. e.g. Komornik-Loreti book (2005)). The 1-d boundary stabilization is also open.

And if anyone thinks that he knows anything, he knows nothing yet as he ought to know.

THANKS!